FAILURE PROBABILITIES OF THREE-TO NINE-PLY CROSS-LAMINATED TIMBER UNDER ROLLING SHEAR LOADING CONSIDERING THE UNDERLYING SERIES SYSTEMS

VERSAGENSWAHRSCHEINLICHKEITEN FÜR DREI- BIS NEUNLAGIGES BRETTSPERRHOLZ BEI ROLLSCHUB-BEANSPRUCHUNG UNTER BERÜCKSICHTIGUNG DER ZUGRUNDELIEGENDEN SERIENSYSTEME

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SUMMARY

The failure probabilities of three- to nine-ply cross-laminated timber (CLT) under rolling shear loading are derived in closed form. The four-point bending test, as defined in EN 16351 for determining the mechanical property “rolling shear strength of the transverse plies”, is used as an example of this type of loading. The transverse plies in the two shear-stressed areas, located between the supports and the force application points, are considered to be stochastically independent components of a series system. These components are assigned a two-parameter logarithmic normal distribution in order to represent the random variable “rolling shear strength”. The parameters of this statistical distribution are obtained from the definable input variables “5% quantile and coefficient of variation”. The shear stress distributions are determined using a simplified approach based on linear-elastic composite theory. Reduction factors are applied to the maximum shear stress in the neutral axis, if the magnitude of the shear stress in the components varies. The failure probability of CLT under load is determined using the cumulative distribution function for the resistance of a series system as well as the simplest probabilistic design model. In this, only the resistance R is a random variable, whereas the load S is deterministic. Based on these simplifying assumptions, the failure probabilities of three- to nine-ply CLT are presented as functions of shear force and ply lay-up. The four failure probabilities are then compared with each other, and their key statistical values are provided. Through factor decomposition, the failure probabilities for seven- and nine-ply CLT are split up

into ply failure probabilities for the inner and outer transverse plies. Finally, the four CLT ply lay-ups are compared with each other for a given shear force using safety indices.

ZUSAMMENFASSUNG

Ausgehend von der Versuchsanordnung des Vierpunkt-Biegeversuchs zur Bestimmung der mechanischen Eigenschaft „Rollschubfestigkeit der Querlagen in Brettsperrholz“ in der Norm EN 16351 werden die Versagenswahrscheinlichkeiten bei dieser Beanspruchung für drei- bis neunlagiges Brettsperrholz in geschlossener Form hergeleitet. Dabei werden sämtliche Querlagen in den beiden schubbeanspruchten Bereichen zwischen den Auflagern und den Krafteinleitungen als stochastisch unabhängige Komponenten eines Seriensystems aufgefasst, denen die zweiparametrische logarithmierte Normalverteilung zur Abbildung der Zufallsvariable „Rollschubfestigkeit“ zugewiesen wird. Die Parameter dieser statistischen Verteilung werden aus den wählbaren Eingangsgrößen „5%-Quantile und Variationskoeffizient“ bestimmt. Die Schubspannungsverteilungen werden vereinfachend gemäß der linear-elastischen Verbundtheorie ermittelt. Treten in den einzelnen Komponenten unterschiedlich hohe Schubspannungen auf, so werden diese durch Abminderungsfaktoren für das Schubspannungsmaximum in der neutralen Faser bestimmt. Um die Versagenswahrscheinlichkeit eines Brettsperrholzaufbaus unter Belastung in Verbindung mit der Wahrscheinlichkeitsverteilung für den Widerstand eines Seriensystems zu erhalten, wird das einfachste probabilistische Bemessungsmodell zugrunde gelegt: Darin ist nur der Widerstand R eine Zufallsvariable, wohingegen die Belastung S als deterministische Größe eingeht. Unter den genannten vereinfachenden Annahmen werden die Versagenswahrscheinlichkeiten von drei- bis neunlagigem Brettsperrholz in Abhängigkeit von der Querkraft und dem Aufbau angegeben. Die vier hergeleiteten Versagenswahrscheinlichkeiten werden einander gegenübergestellt und ihre wesentlichen statistischen Kennwerte miteinander verglichen. Mittels Faktorzerlegung werden die Versagenswahrscheinlichkeiten für sieben- und neunlagiges Brettsperrholz in die Lagen-Versagenswahrscheinlichkeiten für die inneren und die äußeren Querlagen aufgeteilt. Abschließend werden die vier Brettsperrholzaufbauten für eine vorgegebene Querkraft mit Hilfe von Sicherheitsindices bewertet.

1. INTRODUCTION

Cross-laminated timber (CLT) has been a regulated construction product for over a quarter of a century, whose load-bearing behaviour and capacity have also been extensively investigated in combination with mechanical fasteners. Based on the European Assessment Document EAD 130005-00-0304 [1], a large number of European Technical Assessments have been issued. Additionally, German general technical approvals/general type approvals exist for this construction product. However, CLT cannot be CE marked in accordance with the European standard EN 16351 [2], as this has not yet been published in the Official Journal of the European Union. EN 16351 is currently under revision. Nevertheless, the standard is widely used as a technical reference document for this construction product.

All existing building regulations pay special attention to the design of the mechanical property "rolling shear strength of the transverse plies", which is becoming increasingly relevant as the span-to-depth ratio decreases under bending loads perpendicular to the slab plane. In addition to an in-plane shear test, EN 16351 also provides a four-point bending test to determine this mechanical property. When carrying out such four-point bending tests on CLT, the maximum force is reached when rolling shear failure has occurred in one of the transverse plies in the ply lay-up. Furthermore, rolling shear failure only occurs in one of the two areas subjected to shear force between the supports and the force application points (referred to below as areas A and B), while failure in both areas simultaneously is not observed.

This paper aims to derive the failure probabilities of three- to nine-ply CLT as a function of shear force and ply lay-up, given the definable input variables "5% quantile and coefficient of variation" of the rolling shear strength for a transverse ply in areas A or B. The 5% quantile of the rolling shear strength can be chosen based on the transverse ply design (with or without grooves in the boards, with or without narrow-side gluing of the boards) and the coefficient of variation can be taken from empirical values.

Based on the failure patterns briefly described above, the following statistical modelling can be used to derive the aforementioned failure probabilities: All transverse plies subjected to rolling shear in areas A and B are each considered as components of a statistical series system. Unlike parallel, bridge or "*k-out-of-n*" systems, series systems are characterized by failure occurring when one of the components fails. The components are simplistically but conservatively assumed to be stochastically independent. Varying shear stresses acting on the individual components are addressed using reduction factors for the maximum shear stress in the neutral axis. The cumulative distribution functions for the resistance R of the respective series systems are specified using these assumptions.

The simplest of all probabilistic design models is used to calculate the failure probabilities of CLT under an applied load. In this model, only the resistance R is considered as a random variable, while the load S acting on the CLT perpendicular to the slab plane is assumed to be deterministic. Only the failure mode due to rolling shear loading is considered. Analysing multiple failure modes in series systems requires separate consideration.

The derived failure probabilities enable key statistical values to be estimated, such as the 5% quantile, the median value and the coefficient of variation of the shear force capacity for three- to nine-ply CLT. Both input variables can be varied. Given a shear force, the failure probabilities or the safety indices derived from them can be used to compare CLT ply lay-ups with different ply numbers and thicknesses. Through factor decomposition, the failure probabilities of seven- and nine-ply CLT can be split up into the ply failure probabilities of the inner and outer transverse plies.

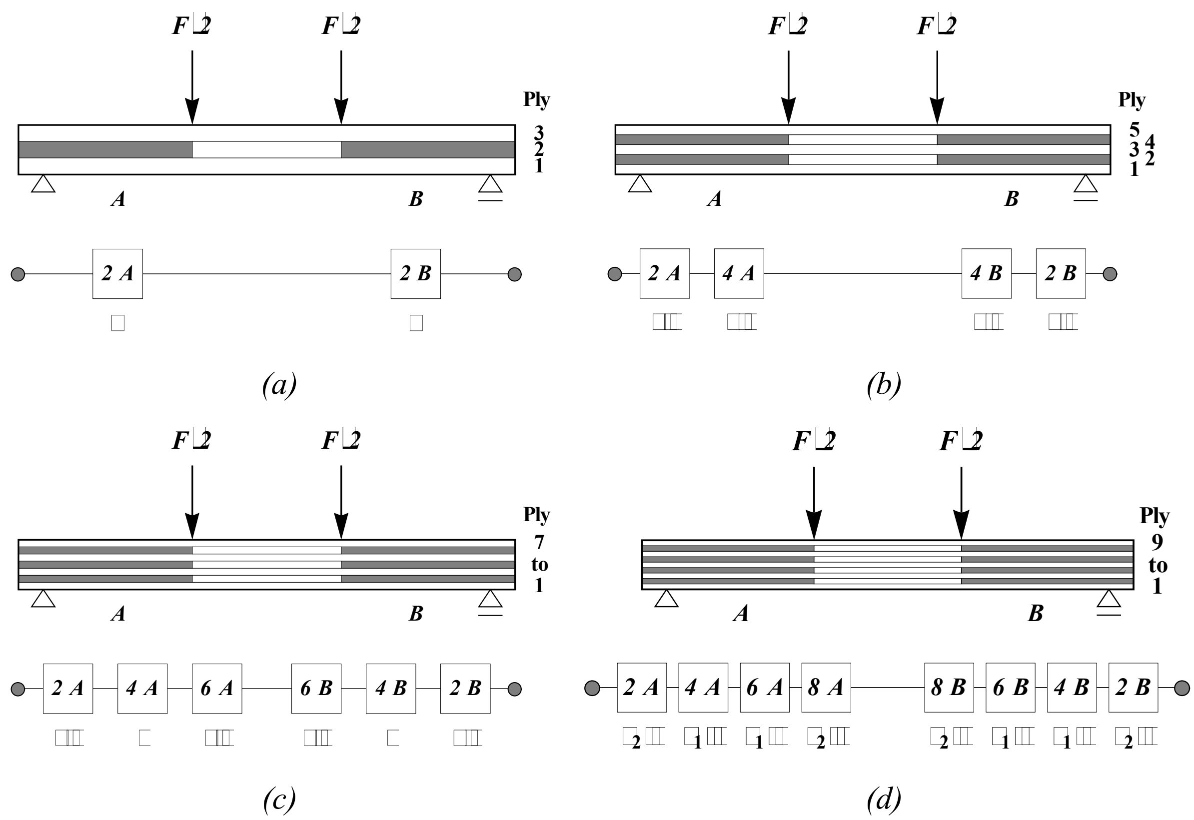
The article is structured as follows: Section 2 provides a more detailed description of the basic approach. Section 3 describes the statistical modelling of series systems, the statistical distributions of their components, and the shear stress distributions. It also describes the probabilistic design model that is applied. Section 4 first presents numerical values, followed by the failure probabilities for three- to nine-ply CLT in closed form. These are then compared with each other. To enable comparison between the different ply lay-ups, evaluations are carried out on CLT with constant ply thicknesses and cross-sectional dimensions of "1" in each case. Section 5 summarizes the results and provides suggestions for further modelling.

2. BASIC APPROACH

The four-point bending test for determining the rolling shear strength of the transverse plies in accordance with EN 16351, Annex C, Figure C.5, is the starting point for deriving the failure probabilities of three- to nine-ply CLT. Fig. 1a shows the corresponding test set-up for three-ply CLT (width $b = 1$, depth $h = 1$, span: $\ell = 9 \cdot h$, distance between supports and force application points: $a = 3 \cdot h$) with constant ply thicknesses $t_i = 1/3 \cdot h$. The areas A and B of the transverse ply between the supports and the force application points are highlighted in grey in this figure. Rolling shear failure in the transverse ply can occur either in area A, between the left support and the left force application point, or in area B, between the right support and the right force application point. The grey areas of the transverse ply are considered as components of a series system of length $s = 2$. This series system is shown graphically as a block diagram below the test set-up in Fig. 1a. The components are designated according to the ply number (counting from the bending tensile zone to the bending compression zone in ascending order) and according to whether they are located in area A or B. Failure of the series system occurs when either component 2 A or component 2 B fails.

Figs. 1b to 1d show the same test set-up for five-, seven- and nine-ply CLT with constant ply thicknesses and cross-sectional dimensions of "1" in each case. The proportions of the transverse plies in areas A and B, in which rolling shear failure may occur, are also shown in grey. These can be understood as series systems of lengths $s = 4$, $s = 6$ and $s = 8$. The corresponding block diagrams are shown graphically below the test set-ups. For five-ply CLT, all four components of the series system are subjected to the same shear stress. This is obtained by multiplying the maximum shear stress in the neutral axis by a reduction factor, κ. However, in the case of seven- and nine-ply CLT, the components of the series systems are subjected to varying shear stresses according to the respective shear stress distribution. These can be determined by multiplying the maximum shear stress in the neutral axis by the reduction factors κ or κ_1 and κ_2 (see the block diagrams in Figs. 1c and 1d).

To statistically represent the rolling shear strength, the components of the series systems shown as block diagrams in Figs. 1a to 1d are assigned a two-parameter logarithmic probability distribution with identical parameters.



Figs. 1a to 1d: Test set-up for determining the rolling shear strength in a four-point bending test for three- to nine ply CLT. The transverse plies subjected to rolling shear stresses are shown in grey. The underlying series systems are depicted as block diagrams below the test set-ups

For low span-to-depth ratios, the determination of the bending and shear stress distributions in CLT typically involves considering the non-linear distribution of horizontal displacements across the depth. In contrast, this paper uses the linear-elastic composite theory to determine the shear stress distributions of the ply lay-ups, since the equation can easily be combined with the chosen probabilistic design model.

To determine the failure probabilities of CLT under load perpendicular to the slab plane, a probabilistic design model is used. The vector of base variables contains only one component: the resistance R of the CLT (i.e., cumulative distribution function of the underlying series system). The load S is assumed to be deterministic. All components of the series systems shown are subject only to rolling shear failure, which is assumed to be "brittle". Other possible failure modes are not considered.

3. STATISTICAL MODELLING

3.1 Series Systems

In a series system, the whole arrangement fails when one component fails. There are two cases that must be distinguished from each other:

- The components of a series system are stochastically independent or
- the components of a series system are stochastically dependent.

If the components are stochastically independent (correlation coefficients between the components $\rho_{ij} = 0$), the failure of one component has no impact on the failure of the remaining components. If the components are stochastically dependent with positive correlation ($\rho_{ij} > 0$), the failure of one component will affect the others. If the components are perfectly stochastically dependent ($\rho_{ij} = 1$), all components will fail simultaneously. Components with negative correlation do not occur in structural systems.

Series systems with stochastically independent components are the most likely to fail, making this the most conservative approach. In series systems, an increasing positive correlation among the components results in a decreasing likelihood of failure. Series systems with stochastically perfectly correlated components are the least likely to fail (see [3]).

Series systems with s stochastically independent components are easy to handle mathematically. The cumulative distribution function $F_{sys}(x)$ of such a series system simply results from the product of the survival probabilities of the individual components:

$$F_{sys}(x) = 1 - \prod_{i=1}^{s} \left(1 - F_i\left(\kappa_j \cdot x\right)\right) \quad (1)$$

where κ_j denote the reduction factors introduced in Section 2 for components in series systems with varying shear stresses.

Series systems with s stochastically dependent components necessitate the integration of s-multivariate probability densities, either analytically or numerically. The computational effort increases with the number of components, and determining the coefficients of the covariance matrix requires further effort. In [4], an exact solution is provided for the special case in which all components are normally distributed, with the same safety index and correlation coefficient.

When testing three-ply CLT in the four-point bending test (see Fig. 1a), rolling shear failure always occurs in the transverse ply either in area A or area B ($P_f = P(A \cup B)$, read: probability of occurrence of event A **or** B). Simultaneous failure in area A and area B ($P_f = P(A \cap B)$, read: A **and** B) cannot be observed. Therefore, for three-ply CLT, the assumption of two stochastically independent components is approximately fulfilled, even though the same shear force in both areas establishes a correlation with regard to rolling shear failure.

When testing nine-ply CLT in the four-point bending test (see Fig. 1d), rolling shear failure occurs in either area A or area B which are assumed to be approximately stochastically independent of each other. However, the four components within each area are stochastically dependent on each other. This is because the failure of one or both of the two inner transverse plies also results in the failure of the outer transverse plies or a proportion of them. This failure pattern indicates correlated component failure. As will be demonstrated later, the components in the outer transverse plies have a much lower ply failure probability due to a lower shear stress acting there. Therefore, the stochastic dependency of the components within areas A or B can also be neglected in statistical modelling.

3.2 Individual Components of Series Systems

The two-parameter logarithmic normal distribution (hereafter referred to as the Log2P normal distribution for short) is chosen to represent the random variable "rolling shear strength" for the components of the series systems shown as block diagrams in Figs. 1a to 1d.

The probability density function (pdf) of the Log2P normal distribution is

$$f(x) = \frac{1}{x \cdot \sigma_y} \cdot \phi_{nor}\left[\frac{ln(x) - \mu_y}{\sigma_y}\right] \; with \;\; \phi_{nor}(z) = \frac{1}{\sqrt{2 \cdot \pi}} \cdot exp\left(-\frac{z^2}{2}\right) \tag{2a}$$

and the cumulative distribution function (cdf) of the Log2P normal distribution is

$$F(x) = \Phi_{nor}\left[\frac{ln(x) - \mu_y}{\sigma_y}\right] \; with \; \Phi_{nor}(z) = \frac{1}{2} \cdot \left[1 + erf\left(\frac{z}{\sqrt{2}}\right)\right] \tag{2b}$$

where $erf(z) = \frac{2}{\sqrt{\pi}} \cdot \int_0^z exp(-x^2)\, dx$ denotes the Gaussian error function.

The parameters of the Log2P normal distribution are the scale parameter μ_y and the shape parameter $\sigma_y > 0$. The quantile function is $x_p = F^{-1}(p) = exp[\mu_y + \sqrt{2} \cdot erf^{-1}(2 \cdot p - 1) \cdot \sigma_y]$. The 5% quantile $x_{05} = F^{-1}(5/100)$ and the 50% quantile $x_{50} = F^{-1}(1/2)$ (i.e., median value) result thus to

$$x_{05} = exp(\mu_y - 1{,}645 \cdot \sigma_y) \tag{2c}$$

and

$$x_{50} = exp(\mu_y) \tag{2d}$$

The coefficients of variation C_{vx} and skewness C_{sx} are used as measures of the spread and the deviation from the normal distribution ($C_{sx} = 0$). The expected value $E(X)$ and the variance $Var(X)$ of the Log2P normal distribution are given by the first moment and second central moment, respectively

$$E(X) = \mu_x = \int_0^\infty x \cdot f(x)\ dx = exp\left(\mu_y + \frac{\sigma_y^2}{2}\right) \tag{2e}$$

and

$$Var(X) = \sigma_x^2 = \int_0^\infty \left(x - E(X)\right)^2 \cdot f(x)\, dx = exp\left(2 \cdot \mu_y + \sigma_y^2\right) \cdot \left[exp\left(\sigma_y^2\right) - 1\right] \tag{2f}$$

Using these statistical quantities, the coefficients of variation C_{vx} and skewness C_{sx} of the Log2P normal distribution take the form

$$C_{vx} = \frac{\sqrt{Var(X)}}{E(X)} = \sqrt{exp\left(\sigma_y^2\right) - 1} \tag{2g}$$

and

$$C_{sx} = \int_0^\infty \left(\frac{x - E(X)}{\sqrt{Var(X)}}\right)^3 \cdot f(x)\, dx = \left[exp\left(\sigma_y^2\right) + 2\right] \cdot \sqrt{exp\left(\sigma_y^2\right) - 1} \tag{2h}$$

As can be seen from Eq. (2h), the Log2P normal distribution always has positive skewness, meaning it is right-skewed.

3.3 Determination of the Shear Stress Distribution

An exact calculation method based on the theory of elasticity for determining stresses in composite materials made of layered anisotropic plies was published by [5]. However, this calculation method is not suitable for determining stresses in CLT in practice due to its mathematical complexity. [6] introduced the shear analogy method for determining stresses in CLT. This method is widely used in conjunction with standard structural analysis computer programs for CLT design. For CLT with up to five plies, the γ-method in accordance with EN 1995-1-1 [7] is also frequently employed.

However, to keep the task tractable in closed form, the following uses the linear-elastic composite theory to determine shear stress distributions. In this case, the equation for the shear stress distribution is given by

$$\tau(z) = \frac{V \cdot S_y(z)}{b \cdot I_y} \tag{3}$$

where V denotes the shear force and $S_y(z) = b \cdot \int_z^{h/2} n(\xi) \cdot \xi d\xi$ and $I_y = b \cdot \int_{-h/2}^{h/2} n(z) \cdot z^2 dz$ denote the stiffness-weighted first and second moments of area, respectively. The ratio $n(z) = E(z)/E_{ref}$ is the normalized ply elasticity distribution or valency function, where $E(z)$ is a piecewise constant function of the modulus of elasticity across the depth and E_{ref} is the reference modulus of elasticity (i.e., the modulus of elasticity of the longitudinal plies).

The shear stress "τ" is subsequently understood to be the maximum shear stress $\tau(0)$ in the neutral axis of the respective ply lay-up. Where varying shear stresses are present in the components of the series systems, the maximum shear stress $\tau = \tau(0)$ in the neutral axis is multiplied by reduction factors, κ_j.

3.4 Probabilistic Design Model

The standard probabilistic design model is the R-S model, in which both the resistance R and the load S are random variables. A detailed description of the R-S model can be found in [8]. Failure of the R-S model occurs when the difference of the random variable $Z = R - S < 0$. If the resistance R and the load S are positive random variables that are stochastically independent, the failure probability P_f of the R-S model is given by the following integral (see, for example, [9])

$$P_f = \int_0^\infty F_R(s) \cdot f_S(s) ds = \int_0^\infty \big(1 - F_S(r)\big) \cdot f_R(r) dr \tag{4a}$$

where $F_R(s)$ is the cdf of the resistance R and $f_S(s)$ is the pdf of the load S.

As previously stated, only the resistance R is introduced as a random variable, while the load S is considered deterministic. The space of the base variables is therefore reduced to the abscissa of the coordinate system, which is divided by the limit state function $g(x)$ into two areas: "failure" ($g(x) < 0$) and "survival" ($g(x) \geq 0$). In this case, the failure probability is given by

$$P_f = \int_{\{x|g(x)<0\}} f_{Sys}(x) dx = \int_0^\infty f_{Sys}(x) \cdot I(g(x) < 0) dx \tag{4b}$$

where $f_{Sys}(x) = dF_{Sys}(x)/dx$ is the pdf of the resistance R of the series system according to Eq. (1) and $I(g(x) < 0)$ is the Boolean indicator function. This function takes the value „1" if the condition in brackets is fulfilled, and the value „0" if it is not.

The limit state function for the rolling shear stress, based on the assumptions in Section 3.3, is given by

$$g(\tau) = R - S = \tau \cdot \frac{b \cdot I_y}{S_y(0)} - V \tag{4c}$$

As can be seen from Eq. (4c), the limit state function is linearly monotonically increasing. This simplifies the calculation of the failure probability to

$$P_f = \int_{x<g^{-1}(0)} f_{sys}(x) dx = \int_0^{g^{-1}(0)} f_{sys}(x) dx = F_{sys}\big(g^{-1}(0)\big) \tag{4d}$$

where $g^{-1}(0)$ is the zero point of the inverse of the limit state function according to Eq. (4c).

4. FAILURE PROBABILITIES

4.1 Assumption of Numerical Values

To numerically evaluate the failure probabilities for three- to nine-ply CLT, homogeneous CLT with lamellas of strength class C24 in accordance with EN 338 [10] (mean value of the modulus of elasticity in bending: $E_{m,0,mean} = 11000\ \mathrm{N/mm^2}$) is considered. The lamellas in the transverse plies of the ply lay-ups are assumed to be in contact with each other without narrow-surface bonding ($E_{m,90,mean} = 0\ \mathrm{N/mm^2}$). The individual plies are glued together crosswise. The cross-sectional dimensions of the CLT ply lay-ups are all set to “1”. The ply thicknesses t_i within the respective CLT ply lay-ups are each constant across the depth (three-ply CLT: $t_i = 1/3 \cdot h$, five-ply CLT: $t_i = 1/5 \cdot h$, seven-ply CLT: $t_i = 1/7 \cdot h$, nine-ply CLT: $t_i = 1/9 \cdot h$).

The 5% quantile of the rolling shear strength is assumed to be $f_{r,05} = 1{,}25\ \mathrm{N/mm^2}$ and the coefficient of variation of the rolling shear strength is assumed to be $C_{vx} = 0{,}15$. When applying the moment method, these two input variables uniquely define the scale parameter μ_y and the shape parameter σ_y of the Log2P normal distribution.

Solving Eq. (2g) yields the shape parameter as a function of the coefficient of variation, $\sigma_y = \sqrt{ln(C_{vx}^2 + 1)}$. Solving Eq. (2c) yields the scale parameter as a function of the 5% quantile and the coefficient of variation, $\mu_y = ln(f_{r,05}) + 1{,}645 \cdot \sigma_y = ln(f_{r,05}) + 1{,}645 \cdot \sqrt{ln(C_{vx}^2 + 1)}$. Substituting the above assumed numerical values yields the parameters of the Log2P normal distribution for each component in the series system: $\mu_y \approx 0{,}469$ and $\sigma_y \approx 0{,}149$.

Where numerical values are given below, they are based on the above assumptions.

4.2 Three-ply Cross-laminated Timber

Three-ply CLT comprises one transverse ply arranged in the middle of the depth, which is therefore subject to the maximum shear stress (reduction factor $\kappa = 1$). Rolling shear failure occurs in one of the two grey areas of the transverse ply (see Fig. 1a). This corresponds to a series system with two independent and identically distributed components (see the block diagram in Fig. 1a). According to Eq. (1), the cdf for the resistance R of the series system is therefore

$$F_{sys}(x) = 1 - \left(1 - F(x)\right)^2 \tag{5a}$$

The failure probability for three-ply CLT, as a function of the shear force V and the ply lay-up, can be obtained by applying Eq. (4d) at the zero point $g^{-1}(0)$ of the inverse limit state function $g^{-1}(\tau) = \frac{S_y(0)}{b \cdot I_y} \cdot (\tau + V)$:

$$P_f(V) = F_{sys}[g^{-1}(0)] = 1 - \frac{1}{4} \cdot erfc\left[\frac{ln\left(\frac{V \cdot S_y(0)}{b \cdot I_y}\right) - \mu_y}{\sqrt{2} \cdot \sigma_y}\right]^2 \quad (5b)$$

where $erfc(z) = 1 - erf(z)$ denotes the complementary Gaussian error function.

As can be seen from Eq. (5b), the probabilistic design model Eq. (4d) simply involves inserting the equation for τ into the cdf of the underlying series system $F_{sys}(x)$.

4.3 Five-ply Cross-laminated Timber

Five-ply CLT comprises two transverse plies, each of which is arranged symmetrically around to the inner longitudinal ply, resulting in a reduced shear stress. Rolling shear failure occurs in one of the four grey areas of the transverse plies (see Fig. 1b). This corresponds to a series system with four independent and identically distributed components (see the block diagram in Fig. 1b). Using the cdf for the resistance R of the series system, $F_{sys}(x) = 1 - \left(1 - F(\kappa \cdot x)\right)^4$, the failure probability for five-ply CLT, as a function of the shear force V and the ply lay-up, is given by

$$P_f(V) = 1 - \frac{1}{16} \cdot erfc\left[\frac{ln\left(\kappa \cdot \frac{V \cdot S_y(0)}{b \cdot I_y}\right) - \mu_y}{\sqrt{2} \cdot \sigma_y}\right]^4 \quad (5c)$$

where $\kappa = S_y(t_3/2 + t_4/2)/S_y(0) = 16/17$ is the reduction factor to be multiplied by the maximum shear stress in the neutral axis.

4.4 Seven-ply Cross-laminated Timber

Seven-ply CLT comprises three transverse plies in total (see Fig. 1c). The inner transverse ply experiences the maximum shear stress, while the two outer transverse plies experience reduced shear stress. This corresponds to a series system of six independent, but not identically distributed components (see the block diagram in Fig. 1c). Using the cdf for the resistance R of the series system, $F_{sys}(x) = 1 - \left(1 - F(x)\right)^2 \cdot \left(1 - F(\kappa \cdot x)\right)^4$, the failure probability for seven-ply CLT, as a function of the shear force V and the ply lay-up, results in

$$P_f(V) = 1 - \frac{1}{64} \cdot erfc\left[\frac{ln\left(\frac{V \cdot S_y(0)}{b \cdot I_y}\right) - \mu_y}{\sqrt{2} \cdot \sigma_y}\right]^2 \cdot erfc\left[\frac{ln\left(\kappa \cdot \frac{V \cdot S_y(0)}{b \cdot I_y}\right) - \mu_y}{\sqrt{2} \cdot \sigma_y}\right]^4 \tag{5d}$$

where $\kappa = S_y(t_4/2 + t_5 + t_6/2)/S_y(0) = 3/4$ is the reduction factor for the outer transverse plies.

The $erfc[*]$-terms in Eq. (5d), multiplied by a constant, represent the survival probabilities of the inner and the outer transverse plies. The symbol " $*$ " stands for the quotients in square brackets in Eq. (5d) without reduction factor. The ply failure probability of the inner transverse ply is $P_{f,inner}(V) = 1 - 1/2^2 \cdot erfc[*]^2$ and the ply failure probability of the two outer transverse plies is $P_{f,outer}(V) = 1 - 1/2^4 \cdot erfc[\kappa \cdot *]^4$. The failure probability $P_f(V)$ can be regained from the ply failure probabilities $P_{f,j}(V)$ using the equation $P_f(V) = 1 - \prod_{j=1}^{m}\left(1 - P_{f,j}(V)\right)$.

4.5 Nine-ply Cross-laminated Timber

Nine-ply CLT comprises four transverse plies in total, which are arranged symmetrically around the longitudinal ply at the center of the depth (see Fig. 1d). The two inner transverse plies experience higher shear stresses than the two outer transverse plies. This corresponds to a series system comprising eight independent, but not identically distributed components (see the block diagram in Fig. 1d). Using the cdf for the resistance R of the series system, $F_{sys}(x) = 1 - \left(1 - F(\kappa_1 \cdot x)\right)^4 \cdot \left(1 - F(\kappa_2 \cdot x)\right)^4$, the failure probability for nine-ply CLT, as a function of the shear force V and the ply lay-up, results in

$$P_f(V) = 1 - \frac{1}{256} \cdot erfc\left[\frac{ln\left(\kappa_1 \cdot \frac{V \cdot S_y(0)}{b \cdot I_y}\right) - \mu_y}{\sqrt{2} \cdot \sigma_y}\right]^4 \cdot erfc\left[\frac{ln\left(\kappa_2 \cdot \frac{V \cdot S_y(0)}{b \cdot I_y}\right) - \mu_y}{\sqrt{2} \cdot \sigma_y}\right]^4 \tag{5e}$$

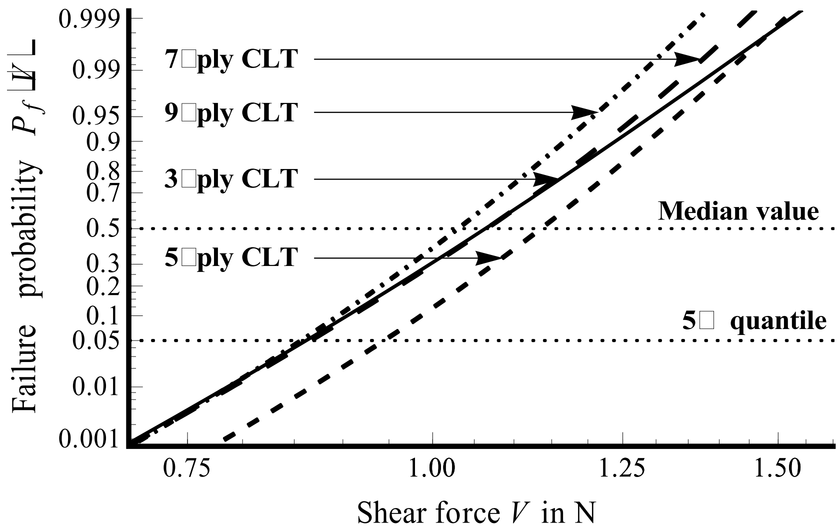


Fig. 2: Failure probabilities of three- to nine-ply CLT with constant ply thicknesses and cross-sectional dimensions of "1" as a function of the applied shear force

where $\kappa_1 = S_y(t_5/2 + t_6/2)/S_y(0) = 48/49$ is the reduction factor for the inner transverse plies and $\kappa_2 = S_y(t_5/2 + t_6 + t_7 + t_8/2)/S_y(0) = 32/49$ is the reduction factor for the outer transverse plies.

The ply failure probability of the inner transverse plies is $P_{f,inner}(V) = 1 - 1/2^4 \cdot erfc[\kappa_1 \cdot *]^4$ and the ply failure probability of the outer transverse plies is $P_{f,outer}(V) = 1 - 1/2^4 \cdot erfc[\kappa_2 \cdot *]^4$, respectively.

4.6 Evaluation of Failure Probabilities and Safety Indices

The failure probabilities specified in Eqs. (5b) to (5e) enable comparisons to be made between the three- to nine-ply CLT with constant ply thicknesses and cross-sectional dimensions of "1" (see Fig. 1a to 1d). Fig. 2 plots these failure probabilities graphically on the Log2P normal probability paper (see Section 4.1 for the numerical values used). In this probability paper, the abscissa and ordinate are scaled so that a Log2P normal distribution appears as a straight line. Table 1 provides some key statistical values for these failure probabilities. These statistical values were calculated numerically in analogy to the equations given in Section 3.2 for the Log2P normal distribution.

As can be seen from Fig. 2 and Table 1, the three- and seven-ply CLT have similar 5% quantiles and median values for shear force. The nine-ply CLT produces slightly lower values. The five-ply CLT differs significantly from all the other ply lay-ups due to its higher 5% quantile and median value for the shear force.

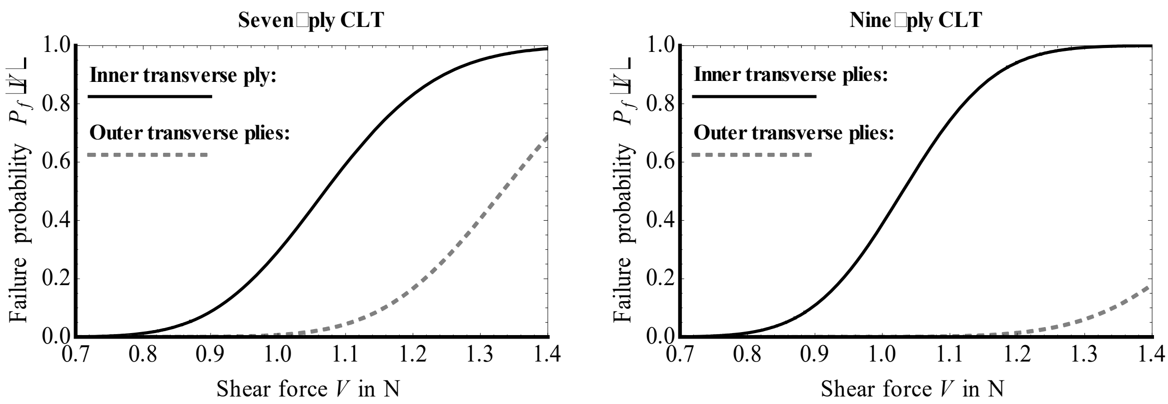
Table 1: 5% quantile V_{05} and median value V_{50} of the shear force as well as coefficients of variation C_{vx} and skewness C_{sx} of failure probabilities $P_f(V)$ for three- to nine-ply CLT with constant ply thicknesses and cross-sectional dimensions of "1"

Ply lay-up	5% quantile	Median value	Coefficient of	
			Variation	Skewness
	V_{05}	V_{50}	C_{vx}	C_{sx}
	N	N	%	-
Three-ply CLT	0,862	1,064	12,3	0,237
Five-ply CLT	0,945	1,136	10,3	0,056
Seven-ply CLT	0,867	1,065	11,5	0,057
Nine-ply CLT	0,857	1,030	10,3	0,045

The coefficient of variation of the Log2P normal distribution for each component was assumed to be $C_{vx} = 15$ %. Due to the series system effect, the coefficient of variation of the failure probabilities tends to decrease as the number of components increases, as can be seen in Table 1. According to Eq. (2h), the coefficient of skewness of the Log2P normal distribution for each component is $C_{sx} = 0{,}453$. As a result of the series system effect, the coefficient of skewness of the failure probabilities also decreases.

Fig. 3a shows the ply failure probabilities for seven-ply CLT, as presented in Section 4.4, plotted against the shear force. As expected, the ply failure probability of the inner transverse ply increases at a significantly lower shear force than that of the outer transverse plies due to higher shear stress. When the failure probability of the inner transverse ply approaches "1", the ply failure probability of the outer transverse plies is only about $\approx\ 0{,}68$.

Fig. 3b shows the ply failure probabilities for nine-ply CLT, considering both the inner and outer transverse plies, as presented in Section 4.5. When the ply failure probability of the inner transverse plies approaches "1", that of the outer transverse plies is only about $\approx\ 0{,}18$. This is significantly lower than the ply failure probability of the outer transverse plies of seven-ply CLT.



(a) *(b)*

Figs. 3a and 3b: Ply failure probabilities of the inner and outer transverse plies in seven- and nine-ply CLT with constant ply thicknesses and cross-sectional dimensions of "1"

The safety index β plays a prominent role in probabilistic design. For normally distributed random variables, a safety index of $\beta = 0$ corresponds to a failure probability of $P_f = 0{,}5$; increasing positive values of the safety index indicate decreasing failure probabilities. Here, the safety index allows for a more illustrative numerical comparison of three- to nine-ply CLT under a given shear force of $V = 1$ N.

The failure probabilities derived above can be converted into safety indices using the equation $\beta \approx -\Phi_{nor}^{-1}\left(P_f(V)\right)$, where $\Phi_{nor}^{-1}(x) = \sqrt{2} \cdot erf^{-1}(2 \cdot x - 1)$ is the inverse standardized normal distribution. The specified ply failure probabilities $P_{f,inner}(V)$ und $P_{f,outer}(V)$ can be converted into ply safety indices β_j using the same equation. The safety index β can be regained from the ply safety indices β_j using the equation $\beta = -\Phi_{nor}^{-1}\left[1 - \prod_{j=1}^{m} \Phi_{nor}(\beta_j)\right]$.

Table 2 shows the safety indices and the ply safety indices (shear force: $V = 1$ N) for three- to nine-ply CLT. As can be seen, the five-ply CLT has the highest safety index by far, while the nine-ply CLT has the lowest. The three- and seven-ply CLT have similar safety indices. The inner transverse plies of the nine-ply CLT have the lowest ply safety index.

Table 2: Safety index β for three- to nine-ply CLT with constant ply thicknesses and cross-sectional dimensions of "1" as well as ply safety indices β_{inner} and β_{outer} (for the five-ply CLT: β_{lower} and β_{upper}) for a given shear force $V = 1\ N$

Ply lay-up	Safety index		
	β	$\beta_{inner\ (lower)}$	$\beta_{outer\ (upper)}$
Three-ply CLT	0,499	-	-
Five-ply CLT	1,164	1,529	1,529
Seven-ply CLT	0,528	0,542	2,464
Nine-ply CLT	0,285	0,286	3,569

An alternative evaluation method arises when the equation for the shear stress distribution according to linear-elastic composite theory is replaced by the variable τ in Eqs. (5b) to (5e). In this case, the failure probability is $P_f(\tau)$ and the three- to nine-ply CLT are compared on the basis of equal shear stress maxima in the neutral axis. Using shear stress τ as a parameter enables the failure probabilities of three- to nine-ply CLT to be parametrically plotted against the failure probability of an individual component. As this evaluation method essentially confirms the results presented so far, it is not pursued further.

5. SUMMARY

Standard EN 16351 specifies a four-point bending test for determining the rolling shear strength of the transverse plies in cross-laminated timber (CLT). Using this test set-up, the failure probabilities of three- to nine-ply CLT due to rolling shear failure in the transverse plies were determined as a function of the shear force and the ply lay-up. In areas subject to rolling shear loading between supports and force application points, the transverse plies were conservatively assumed to be stochastically independent components of a series system. Each component was assigned a two-parameter logarithmic normal distribution to represent the random variable "rolling shear strength". The parameters of this statistical distribution were determined from the definable input variables „5% quantile and coefficient of variation" of this mechanical property. The load acting perpendicular to the slab plane was treated as a deterministic variable.

To simplify the analysis, the shear stress distributions in the three- to nine-ply CLT were determined using linear-elastic composite theory. Where necessary, the maximum shear stress was adjusted to match the actual shear stress using reduction factors. Factor decomposition was employed for seven- and nine-ply CLT to demonstrate that the failure probabilities can be split up into the ply failure probabilities for the highly stressed inner transverse plies and the less stressed outer transverse plies.

Numerical evaluation of the failure probabilities for three- to nine-ply CLT with constant ply thicknesses and cross-sectional dimensions of “1” showed that five-ply CLT has the highest shear force capacity at both the 5% quantile and the median value levels. The other ply lay-ups differed only slightly from each other at the 5% quantile level, while the nine-ply CLT produced the lowest shear force capacity at the median value level. Due to the series system effect, an increasing number of components tends to decrease the coefficients of variation and skewness of the failure probabilities.

The safety indices of three- to nine-ply CLT were evaluated for a given shear force, and the respective ply safety indices were also quantified for five-, seven- and nine-ply CLT. Five-ply CLT had by far the highest safety index (i.e., the lowest failure probability), while the nine-ply CLT had the lowest (i.e., highest failure probability).

The derived failure probabilities can be applied to three- to nine-ply CLT with arbitrary ply thicknesses. The input variables of the statistical model and ply lay-up the can be varied. The influence of these variations will be presented in a further paper. Using the rules of statistical inference, normal-approximation confidence intervals and bands can be computed for the failure probabilities presented.

In statistical modelling, failure probabilities depended solely on the input variables and the ply lay-up. However, this paper did not address two important factors that are essential for the rolling shear strength: the volume effect of the transverse ply thickness and the fullness of the shear stress distributions. While it is in principle possible to incorporate these two influencing quantities using the two-parameter Weibull distribution, the associated mathematical effort is considerable.

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