NUMERICAL STUDY OF MIXED-MODE FRACTURE IN CONCRETE STRUCTURES

NUMERISCHE ANALYSE DER "MIXED-MODE" - BRUCHART VON BETONBAUTEILEN

ANALYSE NUMÉRIQUE DE LA RAPTURE MIXTE DES STRUCTURES EN BÉTON

Joško Ožbolt and Hans-Wolf Reinhardt

SUMMARY

In the present paper, the finite element code based on the microplane model for concrete (program MASA) is used for the analysis of two typical mixed-mode geometries – the notched beam and the double-edge-notched specimen. The local smeared fracture finite element analysis is carried out. As a regularization procedure, the crack band method is used. Comparison between experimental and numerical results shows that the used finite element code is able to realistically predict structural response and crack pattern for mixed-mode fracture of concrete. It is shown that for both investigated geometries mixed-mode fracture mechanism dominates at crack initiation. However, with increase of the crack length mode I fracture becomes dominant and finally specimens fail in failure mode I.

ZUSAMMENFASSUNG

Der vorliegende Bericht beschreibt die numerische Untersuchung von zwei typischen "mixed mode" Geometrien (gekerbter Balken und ein s.g. "Double-Edge-Notched" Probekörper). Die Untersuchung wurde mit dem Finiten-Elementen-Programm MASA, das auf dem "Microplane Model" für Beton basiert, durchgeführt. In der Analyse wurde die verschmierte Rißmethode angewendet. Als Lokalisierungsbegrenzer wurde die Rißbandmethode benutzt. Der Vergleich zwischen den numerischen und experimentellen Ergebnissen zeigt, daß die Analyse in der Lage ist, sowohl die Bruchlast als auch die Rißbildung realistisch darzustellen. Die Rißbildung erfolgte zunächst in der "mixed mode" Bruchart. Mit zunehmender Rißlänge wurde die "mode I" Bruchart zunehmend dominant.

RESUME

Dans article le code de l'élément fini basé sur le modèle microplane en béton (logiciel MASA) est appliqué au deux géométrie typique dans la rupture en mode mixte – la poutre préfissurée et la pièce préfissurée des deux faces. L'analyse de la rupture repartie localité de l'élément fini a été exécuté, couplé au modèle de la bande fissurée. La comparaison entre les résultats de l'essai et l'analyse ont démontré que le logiciel est capable de prédire la réponse de la structure en béton et la forme de la fissure en mode mixte. Il est démontré que dans les deux géométrie étudiée le mécanisme de la rupture en mode mixte domine en début de la rupture. Mais avec le propagation de fissure la rupture en mode I de plus en plus domine et finalement les structures ont été endommagé a la rupture en mode I.

INTRODUCTION

Concrete is a quasi-brittle material that exhibits cracking and damage phenomena. It is today obvious that economical and safe concrete structures cannot be designed without the use of fracture mechanics. Although in the last two decades significant progresses in the field of the application of fracture mechanics in design of concrete and reinforced concrete structures has been made, there are still a number of open questions that need to be solved.

In fracture mechanics terminology there are three different cracking modes defined: (i) mode I – opening mode, (ii) mode II – shearing mode and (iii) mode III – tearing mode. At the macro scale level they describe three independent kinematic movements of the upper and lower crack surface with respect to each other and are sufficient to define all possible modes of crack propagation in an elastic material. Of course, at the micro scale the stress distribution is much more complex and at such a level modes of fracture have no sense. As far as concrete is concerned, mode I is a relatively clear type of crack propagation. On the contrary, mode II and III are complex failure modes, which can hardly be realized in an experiment. In these modes the stress normal to the crack surface need to be approximately zero and only in-plane shear stress should exist. Even when these conditions can be realized, due to the complexity of the concrete structure, over a concrete crack surface a combination of different stresses exist (shear, tension, compression and bending). Although the resulting stress may be in-plane stress (shear), complex stress-strain conditions on a crack surface make the identification of mode II and III fracture parameters extremely difficult. Moreover, the question arises whether in a sense of linear elastic fracture mechanics these two failure modes even exist. The similar complex combination of stresses exists for mode I fracture type as well, however, the stress and strain perpendicular to the crack surface dominate at this fracture type.

In practice most cracks in concrete and reinforced concrete structures result from mode I loading as well as in the combination between mode I and other modes. One of rather frequent combinations is the combination between mode I and mode II, so-called mixed mode fracture. Typical examples are diagonal shear failure of reinforced concrete beams or punching of flat slabs. In the past an, enormous amount of experimental and theoretical work has been done to understand mode I of fracture. On the contrary, such a large number of studies were not devoted to mixed-mode fracture of concrete. For this reasons, nonlinear mixed-mode theories are not yet well developed for concrete like materials [1].

In recent years a significant progress in modeling of concrete like materials for general stress-strain histories has been achieved. Presently available models for concrete can roughly be classified in two categories: (i) Macroscopic models, in which the material behavior is considered to be an average response of a rather complex microstructural stress transfer mechanism and (ii) microscopic models, in which the micromechanics of deformations are described by stress-strain relations on the microlevel. No doubt, from the physical point of view microscopic models are more promising. However, they are computationally extremely demanding. Therefore, in practical applications macroscopic models are used.

Traditionally, macroscopic models are formulated by a total or incremental formulation between the σ_{ij} and ε_{ij} components of the stress-strain tensor, using the theory of tensorial invariants [2][3]. In the framework of the theory there are various possible approaches for modeling of concrete, such as theory of plasticity, plastic-fracturing theory, continuum damage mechanics, endocronic theory and their combinations of various forms. Due to the complexity of concrete these models can not realistically represent the behavior of concrete for general three-dimensional stress-strain histories. Therefore, to formulate a more general and relatively simple model significant effort in further development of the microplane model for concrete has recently been done [4][5].

Cracking and damage can principally be modeled in two different ways: (i) discrete (discrete crack model) or (ii) smeared (smeared crack model). The classical local smeared fracture analysis of materials which exhibit softening (quasi-brittle materials) leads in the finite element analysis to the results, which are mesh dependent [6]. As well known, the reason for this is the localization of strains in a row of finite elements and a related energy consumption capacity that depends on the element size. Consequently, the model response is mesh dependent. To assure mesh independent results, total energy consumption capacity has to be independent of the element size, i.e. one has to regularize the problem by introducing a so-called localization limiter. Currently two different approaches are in use. The first one is the relatively simple crack band method [7] and the second ones are the so-called higher order methods: Cosserat continuum [8] and nonlocal continuum approaches of integral type [9][10] or gradient type [8]. Compared to the crack band method the higher order procedures are more general but rather complex and related to different problems (require extremely fine meshes, problems with boundaries, difficult to identify the material parameter and other). Mesh independent results can alternatively be obtained by the use of the discrete crack approach [11]. The main drawback of this approach is the need for continuous re-meshing, which is a rather complex and time consuming procedure. Moreover, some stress-strain situations (for instance compression) are difficult to model in a discrete sense. Therefore, the smeared crack approach is a more general approach, especially for practical engineering applications.

To better understand mixed-mode fracture of concrete and to see whether it is possible to model it by employing a constitutive law, which is calibrated using only mode-I fracture test data, the numerical analysis of two typical mixed-mode geometries is carried out. Moreover, it is investigated whether the local smeared crack finite element code based on the crack band theory is able to realistically predict relatively complex mixed-mode failure of concrete members. Studied are a beam tested by [12] and a Double-Edge-Notched (DEN) geometry tested by Nooru-Mohamed [1]. As a material constitutive law the microplane model for concrete, recently proposed by Ožbolt et al. [5] is employed. As a localization limiter the crack band theory [7] is used.

FINITE ELEMENT MODEL

General

The finite element code (MASA) employed in the present study can be applied for the nonlinear finite element (FE) analysis of concrete and reinforced concrete structures [13]. It is based on the microplane material model and a smeared crack concept. As regularization procedures the standard or improved crack band approach (stress relaxation method) can be used. Alternatively, the nonlocal integral approach can be employed as well. The concrete is discretized by four-node quadrilateral elements (plane analysis) or by eight-node brick elements (three-dimensional analysis). The reinforcement is represented by truss or beam elements. Optionally, it can also be modeled in a smeared way, i.e. smeared inside a row of concrete elements. Besides these standard elements, special linear or nonlinear contact elements are available as well. The analysis is incremental with a solution procedure based on the secant or constant stiffness method.

Constitutive law for concrete – microplane material model

In the microplane model the material is characterized by a relation between the stress and strain components on planes of various orientations. These planes may be imagined to represent the damage planes or weak planes in the microstructure, such as contact layers between aggregates in concrete (see Figure 1). In the model the tensorial invariance restrictions need not to be directly enforced. Superimposing the responses from all microplanes in a suitable manner automatically satisfies them. G.I. Taylor advanced the basic concept behind the microplane model in 1938 [14]. Later the model was extended by Bažant and co-workers for modeling of quasi-brittle materials which exhibit softening (Bažant and Prat, 1988; Ožbolt and Bažant, 1992; Carol et al., 1992; Ožbolt et al., 2000).

The recently proposed version of the microplane model for concrete is based on the so-called relaxed kinematic constraint concept [5]. In the model the microplane (see Figure 1) is defined by its unit normal vector of components n_i. Normal and shear stress and strain components (σ_N, σ_{Tr}, ε_N, ε_{Tr}) are considered on each plane. Microplane strains are assumed to be the projections of the macroscopic strain tensor ε_{ij} (kinematic constraint). Based on the virtual work approach, the macroscopic stress tensor is obtained as an integral over all possible, in advance defined, microplane orientations (Ω denote the surface of the unit sphere):

$$\sigma_{ij} = \frac{3}{2\pi}\int_{\Omega} \sigma_N n_i n_j d\Omega + \frac{3}{2\pi}\int_{\Omega} \frac{\sigma_{Tr}}{2}(n_i \delta_{rj} + n_j \delta_{ri})\, d\Omega \tag{1}$$

To realistically model concrete, the normal microplane stress and strain components have to be decomposed into volumetric and deviatoric parts ($\sigma_N = \sigma_V + \sigma_D$, $\varepsilon_N = \varepsilon_V + \varepsilon_D$; see Figure 1), what leads to the following expression for the macroscopic stress tensor:

$$\sigma_{ij} = \sigma_V \delta_{ij} + \frac{3}{2\pi}\int_{\Omega} \sigma_D n_i n_j d\Omega + \frac{3}{2\pi}\int_{\Omega} \frac{\sigma_{Tr}}{2}(n_i \delta_{rj} + n_j \delta_{ri})\, d\Omega \tag{2}$$

For each microplane component, the uniaxial stress-strain relations are assumed as:

$$\sigma_V = F_V(\varepsilon_{V,eff}) \; ; \quad \sigma_D = F_D(\varepsilon_{D,eff}) \; ; \quad \sigma_{Tr} = F_{Tr}(\varepsilon_{Tr,eff}) \tag{3}$$

where F_V, F_D and F_{Tr} are the uniaxial stress-strain relationships for volumetric, deviatoric and shear components, respectively. From known macroscopic strain tensor, the microplane strains are calculated based on the kinematic constraint approach. However, in (3) only effective parts of these strains are used to calculate microplane stresses. Finally, the macroscopic stress tensor is obtained from (2). The integration over all microplane directions (21 directions) is performed numerically.

To model concrete cracking for any load history realistically, the effective microplane strains are introduced. They are calculated as:

$$\varepsilon_{m,eff} = \varepsilon_m \psi \tag{4}$$

where subscript m denotes the corresponding microplane components (V, D, T_r) and Ψ is a so called discontinuity function. This function accounts for discontinuity of the macroscopic strain field (cracking) on the individual microplanes. It "relaxes" the kinematic constraint, which is in the case of strong localization of strains physically unrealistic. Consequently, in the smeared fracture type of the analysis the discontinuity function Ψ enables localization of strains, not only for tensile fracture, but also for dominant compressive type of failure.

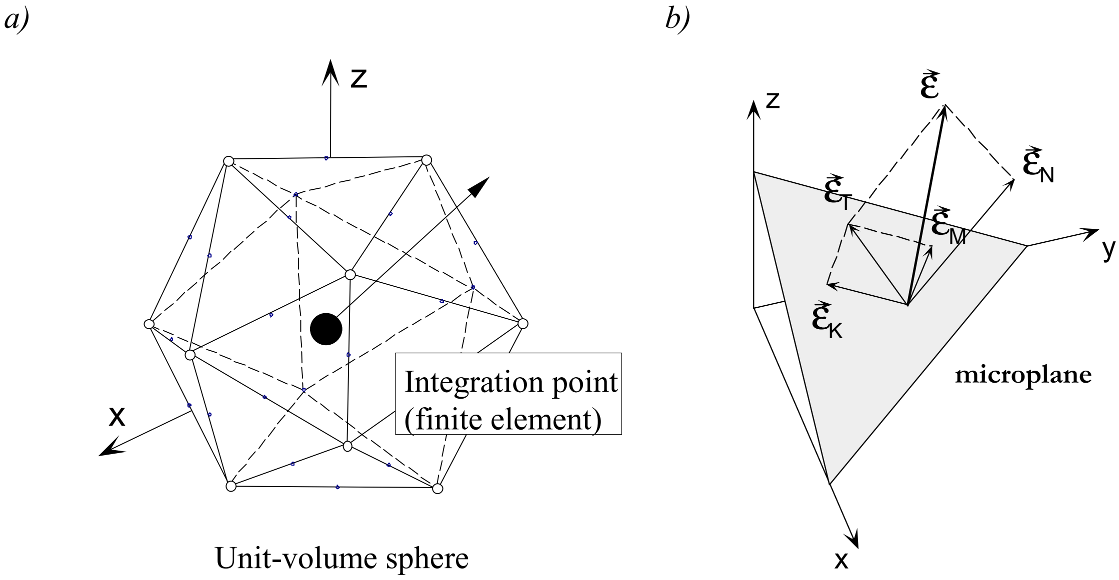


FIGURE 1. The concept of the microplane model:
a) discretization of the unit volume sphere for each finite element integration point (21 microplane directions)
b) microplane strain components

Localization limiter - crack band method

The main assumption of the crack band method is that damage (crack) is localized in a row of single finite elements. To assure a constant and mesh independent energy consumption capacity of concrete (concrete fracture energy G_F) the constitutive law needs to be modified such that:

$$G_F = A_f\, h = const. \tag{5}$$

where A_f = the area under the uniaxial tensile stress-strain curve and h = average element size (width of the crack band). Principally, the same relation is valid for uniaxial compression with the assumption that the concrete compressive fracture energy G_C is a material constant:

$$G_C = A_{fc}\, h = const. \tag{6}$$

in which A_{fc} = area under the uniaxial compressive stress-strain curve. It is assumed that G_C is approximately 100 times larger than G_F ($G_C \approx 100\ G_F$). From (5) and (6) is obvious that the constitutive law for concrete needs to be adapted to the element size.

Willam's test - rotation of principal directions

To check whether the proposed model predicts consistent solution for tensile dominant load with significant rotations of principal stresses, what is relevant for modeling of mixed-mode fracture, Willam's test has been performed [15]. In this test, uniaxial tension is applied first in the x direction (plane stress state), reaching the onset of tensile cracking. Subsequently strain increments are prescribed to all degrees of freedom proportionally to $\Delta\boldsymbol{\varepsilon} = [\Delta\varepsilon_{xx}, \Delta\varepsilon_{yy}, \Delta\gamma_{xy}]^T = [0.50, 0.75, 1.00]^T$. This implies increments of tensile strain for both principal axes, accompanied by a rotation that reaches asymptotically the value of 38^0, measured from the x direction. In the present example the used material parameters were: Young's modulus E = 32000 MPa, Poisson's ratio $\nu = 0.2$, tensile strength $f_t = 3.0$ MPa, uniaxial compressive strength $f_c = 38.0$ MPa and fracture energy $G_F = 0.11$ N/mm (assumed crack band $h = 60$ mm). The evolution of σ_{xx} and σ_{xy} is shown in Figure 2. For comparison, uniaxial stress-strain response is also plotted. Compared to the uniaxial tensile response, the multidirectional damage reduces the post-peak capacity in x direction and for large positive strains (tension) all stresses correctly reduce to zero.

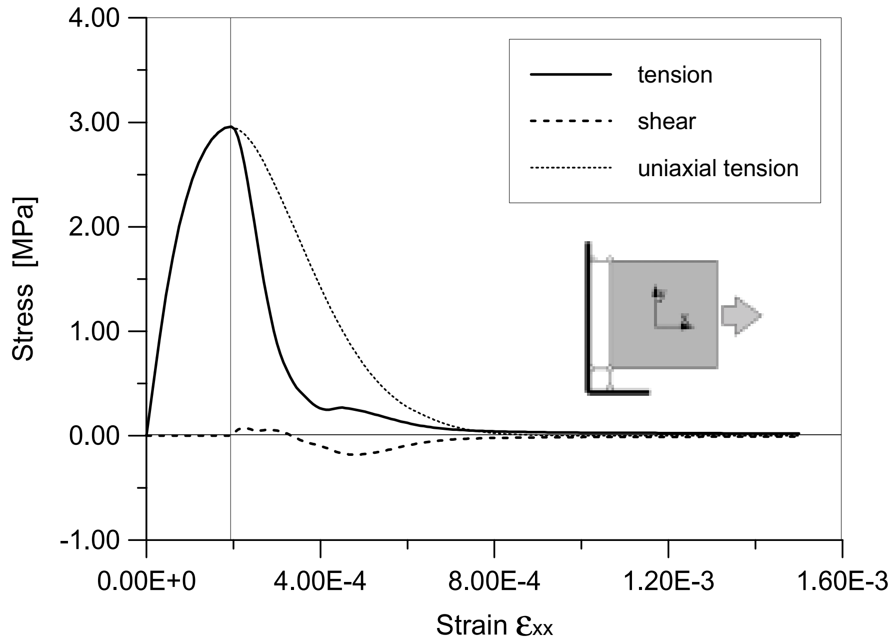


FIGURE 2. Willam's test – rotation of principal axis and comparison with uniaxial tension [15].

NUMERICAL STUDIES OF MIXED-MODE FRACTURE

Mixed-mode fracture of concrete is studied for two different concrete geometries. First is analysed a single notched beam which was for concrete tested by Arrea and Ingraffea [12]. Furthermore, so-called push-off type of the specimen, used by Reinhardt et al. [16], is investigated. The originally proposed geometry is modified because of experimental reasons and because shear in originally proposed push-off specimen is not clear enough. Therefore, the Double-Edge-Notched (DEN) specimen, tested by Nooru-Mohamed [1], is studied. Both specimens are analysed by the use of the above presented microplane model (M2-O, [5]). As a regularization procedure, the crack band method is performed.

Single-edge-notched beam

The geometry and the loading arrangement of the single edge notched beam is shown in Figure 3 [12]. For this geometry the tip of the crack propagates in the mixed-mode stress field. By changing the load and the reaction point the stress field could be varied from mode I to mode II. The chosen loading system yields to the high K_{II}/K_I ratio at the crack tip of the notch (K = stress intensity factor). As the crack propagates from the notch, the above ratio decreases and mode I stress intensity factor becomes dominant. In past the above geometry has been numerically investigated by various authors using different approaches [1]. The objective results could be obtained by micro-mechanical simulations or by using discrete crack approach. However, the smeared crack models often led to residual stresses and to incorrect crack path.

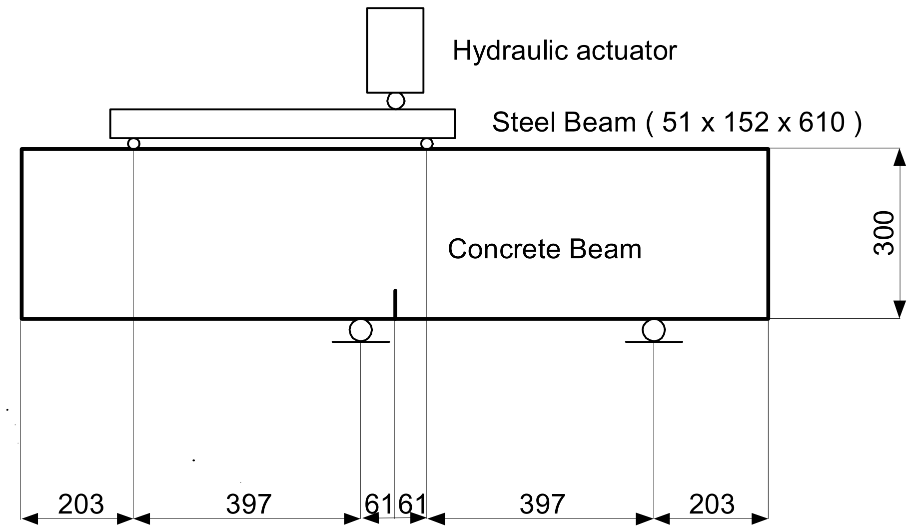


FIGURE 3. Test set-up of the investigated beam geometry, dimensions in mm.

The finite element discretization is shown in Figure 4. Quadrilateral plane elements with four integration points are used. In the analysis the load was applied by controlling the CMSD (Crack Mouth Sliding Displacement). The used material parameters were the same as in the experiment, i.e. Young's modulus E = 30000 MPa, Poisson's ratio $\nu = 0.18$, tensile strength $f_t = 3.5$ MPa and concrete fracture energy $G_F = 0.14$ N/mm.

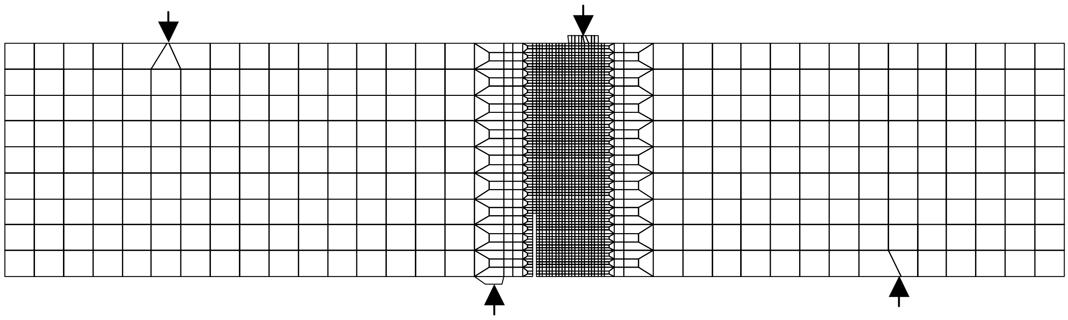


FIGURE 4. The finite element discretization of the beam test specimen.

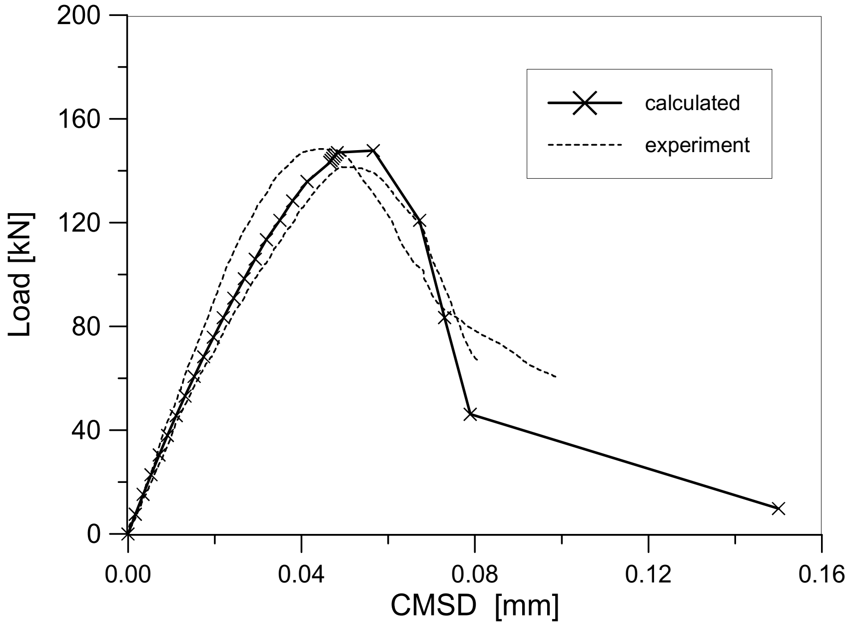


FIGURE 5. Calculated and measured load-CMSD curves.

The calculated and in the experiment measured total load-CMSD curves are plotted in Figure 5. The comparison between numerical and experimental results is reasonably good. Figure 6 shows the calculated crack pattern (maximal principal strains) and in the experiment observed typical crack pattern. As can be seen, the agreement is very good, i.e. the same as in the experiment the final crack tip falls right to the loading plate. The distribution of maximal principal stresses at failure is shown in Figure 7. As can be seen, along the crack path there are no residual stresses.

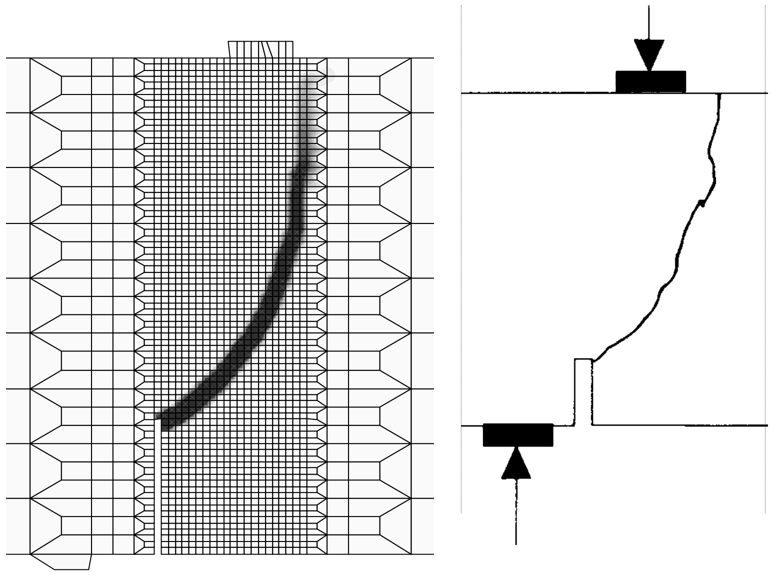


FIGURE 6. Crack pattern: a) experiment and b) analysis (dark zone = maximal principal strains).

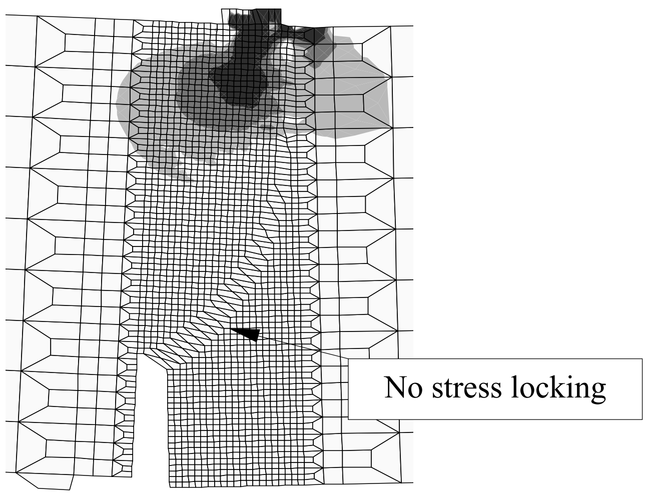


FIGURE 7. Distribution of maximal principal stresses at failure – deformed mesh (black zones = 3 MPa, white zones = 0 MPa).

Figure 7 shows the deformed mesh of the crack region at termination of the analysis. The elements close to the notch tip are distorted in both, vertical and horizontal directions. This indicates mixed-mode condition at initiation of the crack (softening regime). The stress-strain distribution at notch tip is similar to that observed when following the loading path according to Willam's test [15], i.e. during the entire load history the principal strain-axis rotate and the crack opens in direction perpendicular to maximum principal stress. More the crack propagates, less distorted are the elements at the crack tip. At final load stage they are deformed nearly only into the horizontal direction (see Figure 7). This means that mixed-mode fracture, observed at crack initiation, degenerates to pure mode-I fracture at termination of the analysis (failure).

Double edge notched specimen

The Double-Edge-Notched specimen tested by Nooru-Mohamed [1] is analysed using the presented finite element code. The specimen geometry and the test set-up are shown in Figure 8. The specimen was first loaded by shear load S. Subsequently, for constant shear load S, the vertical tensile load T was applied up to failure. The load control procedure was applied by moving of the upper loading platens in horizontal and vertical direction, respectively. The rotation of the loading platens was restricted. During the application of the horizontal load S, the vertical load was kept zero (T = 0). By subsequent tensile loading the shear force was kept constant. The bottom (support) platens were fixed and, the same as the upper (loading) platens, glued to the surface of the specimen. Two case studies are carried out, i.e. for S_{max} = 5 kN and for S_{max} = 10 kN. The finite element discretization is performed by the use of the three-dimensional eight node solid elements with eight integration points (see Figure 9). The width of the finite element model was 5 mm (the actual width of the specimen was 50 mm). The material properties are taken as: Young's modulus $E = 32800$ MPa, Poisson's ratio $\nu = 0.2$, tensile strength $f_t = 3.0$ MPa, uniaxial compressive strength $f_c = 38.4$ MPa and concrete fracture energy $G_F = 0.11$ N/mm.

The calculated and in the experiment measured tensile load versus average normal displacement δT (δT = ($\delta_{MM'}$- $\delta_{NN'}$)/2) are for both load histories shown in Figure 10a. The relation between average normal δT and average shear δS (δS = $\delta_{PP'}$) displacements are plotted in Figure 10b. As can be seen, the agreement between calculated and measured data is reasonably good. The same as in the experiment, the tensile resistance decreases if shear load increases. The calculated resistance overestimate the test data. The reason can be the chosen shape of the tensile softening curves, which are close to the peak resistance possibly to ductile. The another reason could due to the fact that only one row of a three-dimensional elements (thickness of 5 mm) is used in the analysis and no modeling of the complete specimen (thickness of 50 mm) is performed.

Nevertheless, the numerical results are not optimized to the test data, i.e. for a given macroscopic properties of concrete only one numerical analysis is performed.

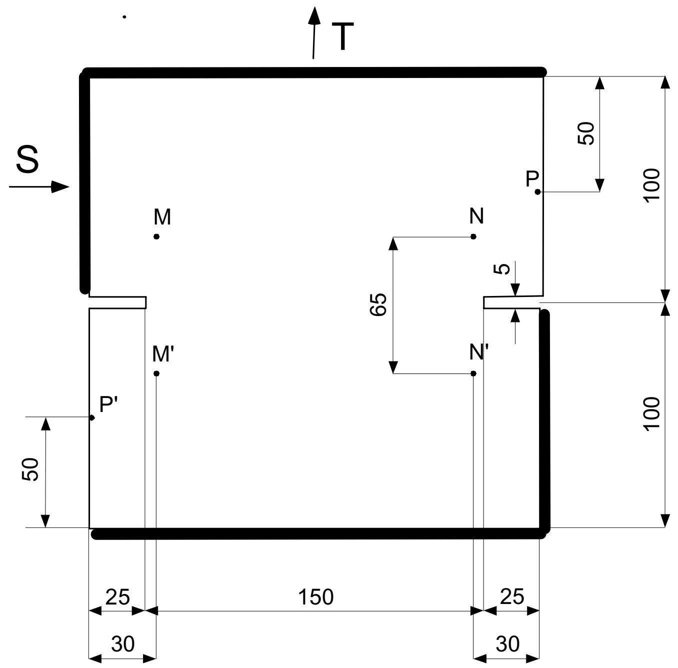


FIGURE 8. Geometry of the DENS specimen.

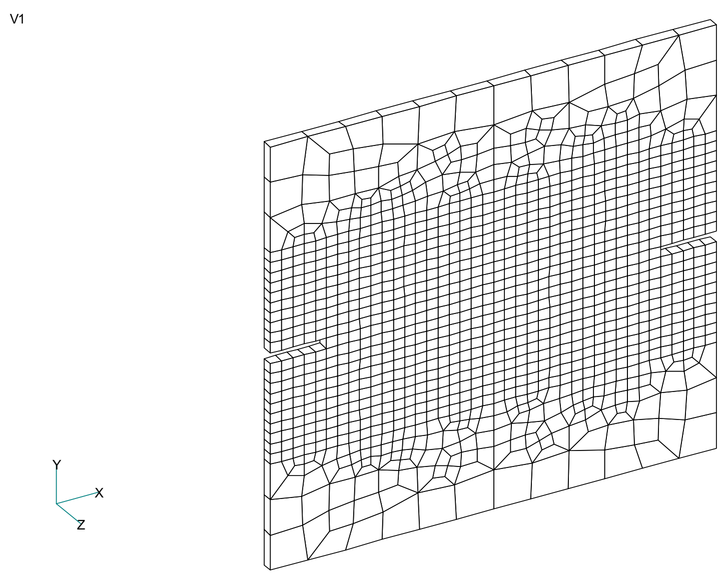


FIGURE 9. Finite element discretization.

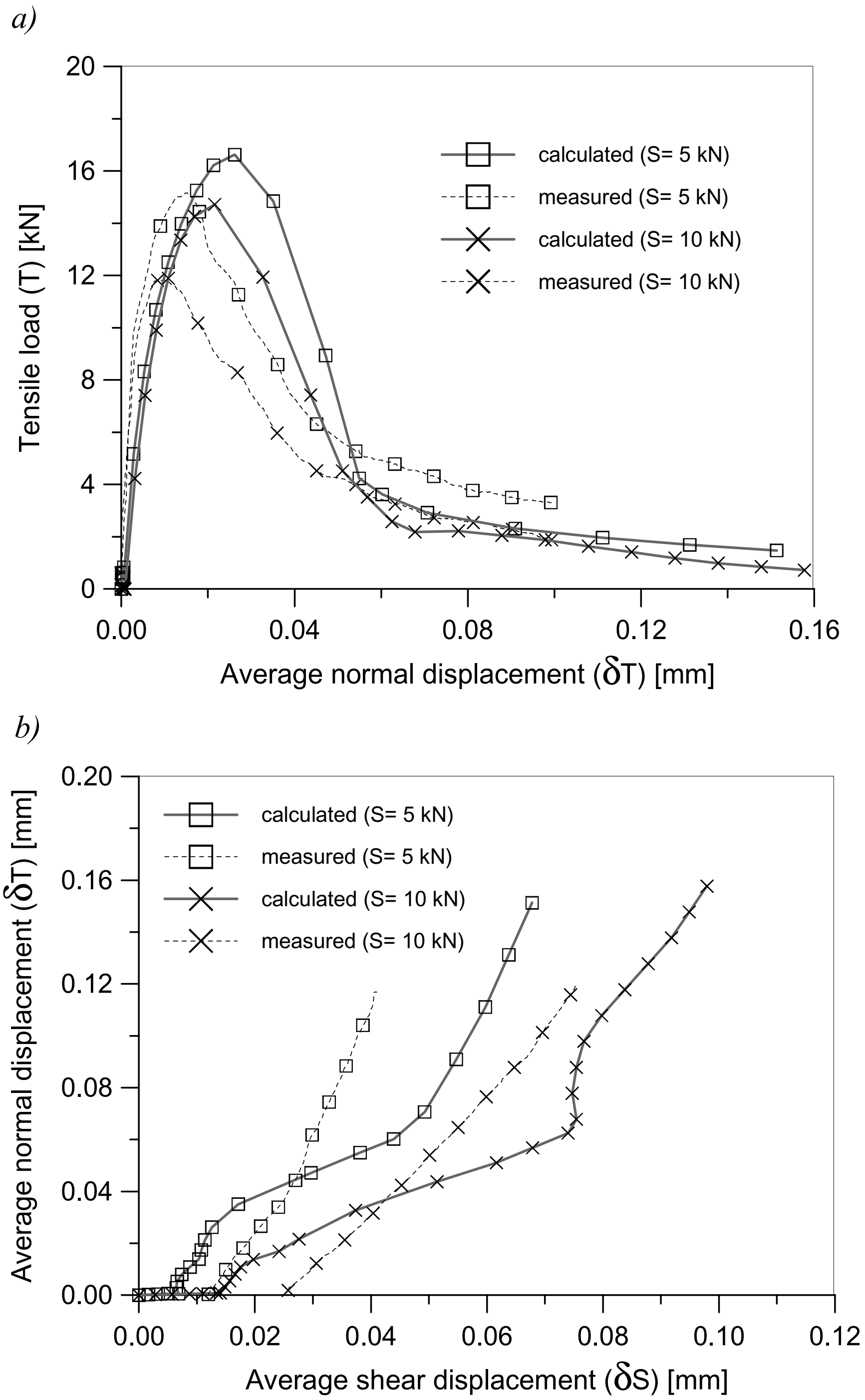


FIGURE 10. Calculated and measured load-displacement curves: a) tensile load as a function of the average normal displacement and b) average normal displacement versus average shear displacement.

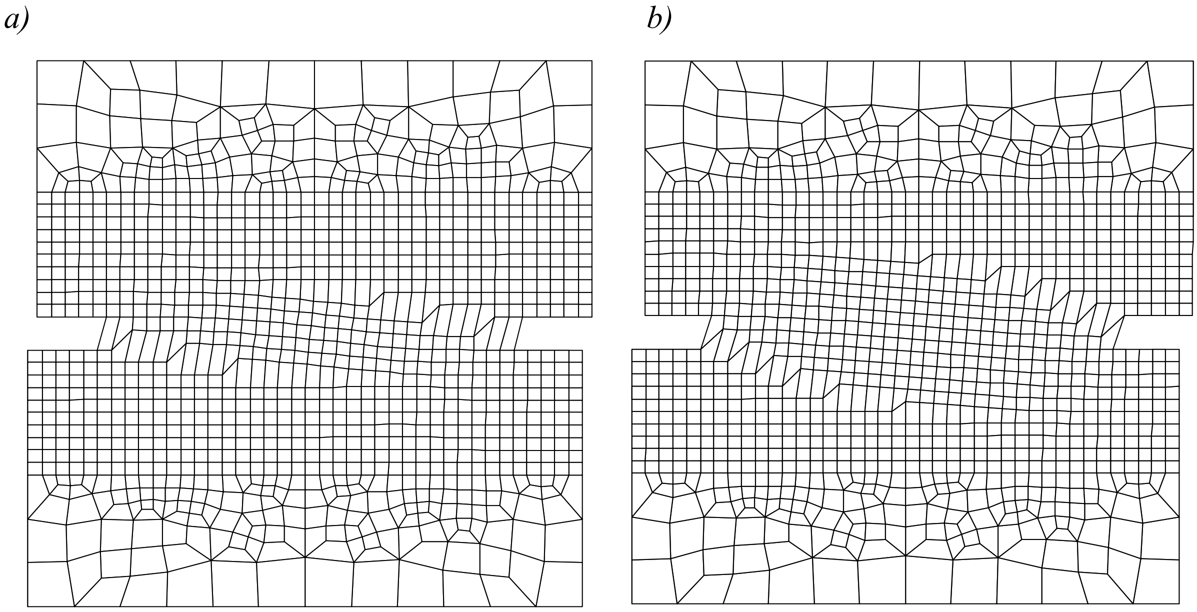
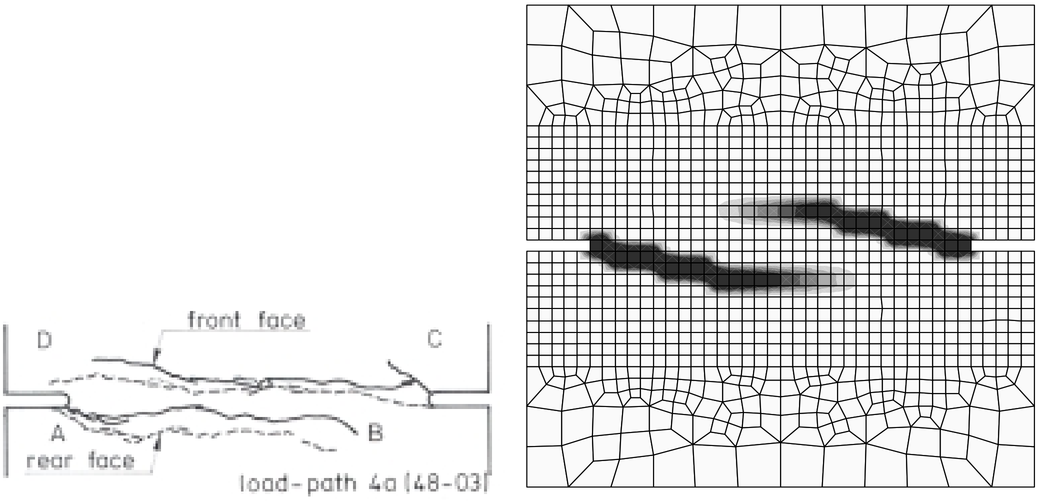


FIGURE 11. Distorted meshes at termination of the analysis:
a) for S_{max} = 5 kN and
b) for S_{max} = 10 kN.

The distorted mesh at termination of the analysis is for both load histories shown in Figure 11. The corresponding crack patterns (maximal principal strains) are shown in Figure 12. For comparison, the crack patterns obtained in the experiment are shown as well. It can be seen that the present finite element code is able to correctly predict the crack propagation for mixed-mode fracture, i.e. the calculated and observed crack patterns are for both load histories almost identical.

Similar as for the beam specimen, the mixed-mode failure mechanism is dominant at initiation of the crack (see distored finite elements at the notch tip in Figure 11). It is more pronounced when the shear force is larger (compare Fig 11a and 11b). Consequently, for larger shear force (mode II), tensile resistance decreases. However, at termination of the analysis mode-I dominates. This can be seen from Figure 11 which shows that at the crack tip the elements are deformed only in direction of tensile load (vertical direction). At this stage, the shear force, which is for the entire tensile load history constant, is transferred from the loading plate to the vertical support plate over a compressive strut which forms between two cracks (see Figure 13).

a)



b)

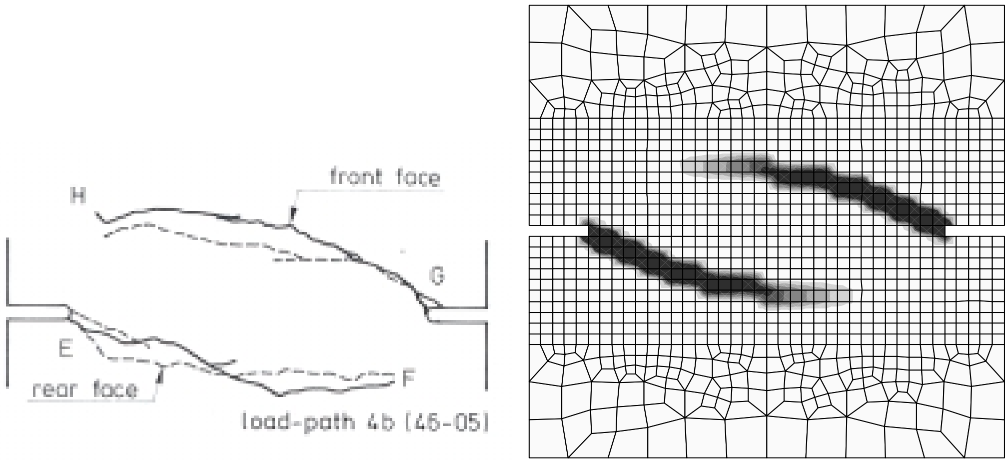


FIGURE 12. Crack patterns observed in the experiment and in the analysis: a) for S_{max} = 5 kN and b) for S_{max} = 10 kN.

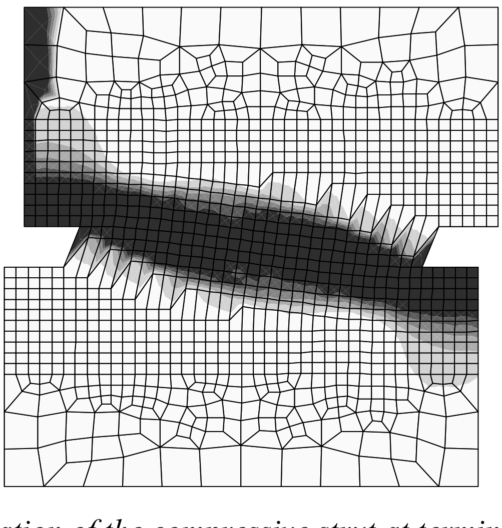


FIGURE 13. Formation of the compressive strut at termination of the analysis (dark zone = compressive zone).

CONCLUSIONS

In the present paper, the finite element code based on the microplane model for concrete and local smeared fracture concept is used for the analysis of two typical mixed-mode geometries. As a regularization procedure, the crack band method is used. Based on the numerical results and their comparison with test data, the following conclusions can be drawn: (1) It is demonstrated that the used local continuum finite element code is able to realistically predict structural response and crack pattern for mixed-mode fracture of concrete; (2) Although the local finite element code is used, no sensitivity with respect to the orientation of the finite elements is observed. Moreover, the analysis shows no stress locking; (3) For both investigated geometries, mixed-mode fracture mechanism dominates at the crack initiation, however, with increase of the crack length mode I becomes dominant. Finally, both specimens fail almost in pure mode I fracture.

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