

**BASIS FOR THE DETERMINATION OF THE ELASTIC PARAMETERS IN CONCRETE
REFERRING TO MEASUREMENTS OF BODY AND SURFACE WAVE VELOCITIES**

**GRUNDLAGEN FÜR DIE BESTIMMUNG DER ELASTISCHEN
PARAMETER IN BETON AUS MESSUNGEN DER RAUM- UND OBERFLÄCHEN-
WELLENGESCHWINDIGKEITEN**

**ÉLÉMENTS POUR LA DÉTERMINATION DES PARAMÈTRES ÉLASTIQUES EN
BÉTON PAR LA VITESSE DES ONDES SPATIALES ET SUPERFICIELLES**

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SUMMARY

Within the context of a research project, FMFA is studying green (unset) concrete. The principal aim is the determination of the elastic parameters (i. e. Young's modulus, rigidity modulus, Poisson's ratio) during the hardening of concrete. The measurements shall be executed dynamically, this means using ultrasonic pulses. The following paper shows the fundamental possibility to determine the elastic parameters measuring only longitudinal and surface wave velocities. From the point of view of form, this deduction is only valid for ideal-elastic solid objects. A first approach to transfer this theory to visco-elastic media is described at the end.

ZUSAMMENFASSUNG

Im Rahmen eines Forschungsprojektes (DAfStb) befaßt sich die FMFA mit der Untersuchung von Frischbeton. Ziel ist vor allem die Bestimmung der elastischen Parameter (wie E-Modul, Scher-Modul und Poissonzahl) in der Erhärtungsphase des Betons. Dabei sollen die Messungen dynamisch mit Hilfe von Ultraschallsignalen erfolgen. Im vorliegenden Artikel wird gezeigt, daß es prinzipiell möglich ist, die elastischen Parameter lediglich durch die Messung von

Kompressions- und Oberflächenwellen-Geschwindigkeit zu bestimmen. Formal gilt diese Ableitung jedoch nur für ideal-elastische feste Körper. Erste Ansätze für die Übertragung auf visko-elastische Medien werden beschrieben.

RÉSUMÉ

En rapport avec un projet de recherche, la FMPA s'occupe de l'étude de béton verd. Le but à ce propos est surtout la spécification des paramètres élastiques (p. e. le module d'élasticité, le module de rigidité et le nombre Poisson) pendant l'endurcissement du béton. Il faut que les mesures soient exécutées dynamiquement, c'est-à-dire à l'aide des ondes ultra-sonores.

Il est montré dans cet article que c'est en général possible de déterminer les paramètres élastiques par des mesures de la vitesse des ondes de compression et de surface. Cependant, cette déduction est valide seulement pour objets idéal-elastiques. Une première approche de l'application aussi à des matériaux visco-élastiques est décrite à la fin.

KEYWORDS: Non-destructive testing, ultrasonics, Young's modulus, Poisson's ratio, green concrete, continuum mechanics.

1. INDEX OF ABBREVIATIONS AND MATHEMATICAL SYMBOLS

The use of abbreviations in accordance with (geo-) physical conventions facilitates the equations to be displayed with more brevity. A listing of the abbreviations used in this article follows. In addition, we used the Nabla operator

∇ (standing for the partial differentiation with respect to Cartesian coordinates) as well as the Laplace operator Δ (for the second derivative) according to reference [3].

v_p :	longitudinal wave velocity (compression wave, P-wave)
v_s :	transverse wave velocity (shear wave, S-wave)
v_r :	Rayleigh or surface wave velocity
ξ :	proportional factor between v_s and v_r
x, y, z :	Cartesian coordinates
ϵ :	Young's modulus (adiabatic)
\mathbf{T}_{ik} :	stress tensor
\mathbf{D}_{ik} :	deformation tensor
\mathbf{E}_{ijkl} :	elastic stiffness tensor
θ :	dilatation (trace of the deformation tensor)
δ_{ik} :	Kronecker symbol
λ, μ :	Lamé's constants
ρ :	density of medium
f_i :	applied force (of space)
η :	dynamic viscosity
σ :	Poisson's ratio (adiabatic)
ω :	angular frequency
k :	wave number
u :	displacement
t :	time
Index s:	transverse part
Index p:	compression part

While the position of a point in the given system of coordinates is represented by a vector \vec{r} , the displacement of a point, resulting from a deformation, shall be indicated by $\vec{u} = \vec{r}' - \vec{r}$. Using index notation x_i for the components of \vec{r} ,

we may write $u_i = x_i' - x_i$. Generally, we used Einstein's summation convention, in which a repeated index denotes summation over the range of the index. As usual, the symbols for summation over vector and tensor indices may be omitted.

2. INTRODUCTION

Visco-elastic media are generally investigated using rheological models. The classical model for the rheological behaviour of green concrete is the so-called Bingham model. The general Bingham model describes the plastic behaviour of a body assembled as a combination of a linear-elastic body (Hooke, spring) with a non-frictionless liquid, and a rigid-plastical body. (St. Venant).

For the determination of the elastic parameters of visco-elastic media there are a number of measuring methods that in principle can be divided up into static (or quasi-static) and dynamic methods. At this point we want to study exclusively dynamic methods, to be precise especially the ones basing on the investigation of wave propagation. The direct applicability of the existing ultrasonic-methods, however, is limited by two aspects:

- I.) The theory of wave propagation in linear-elastic media is already a solved problem [1, 2]. Unfortunately, green concrete shows no linear behaviour (Bingham model). That means, it has to be tested for quantitative measurements, how much the theory is influenced by a change of the state of aggregation of concrete.
- II.) For reasons of the high attenuation and of the liquid state of green concrete during the first time, there are considerable difficulties in investigating the

wave propagation of P-, S-, and surface waves due to the technique of measuring. As even the determination of the S-wave-arrival-times in solid concrete represents a great problem, it doesn't seem to be feasible in green concrete.

Before carrying out US-measurements of green concrete in laboratory researches (I), we want to focus on point II. We will try to find an answer to the question, if it is at all possible to determine the elastic parameters (i. e. Young's modulus, Poisson's ratio, and so on) by measuring only the longitudinal and the surface wave velocity. While deducing the physical relations, we made some simplifications and assumptions (like elasticity, homogeneity, isotropy, and so on) that have to be well-understood for finding a transition to visco-elastic media later on. That is why the development of the wave equations and especially the decoupling of body and surface waves are presented very detailed (note that there are several different ways of deducing the equations developed in this article). Basing on these equations, in ch. 9, we will show a way to introduce inelasticity into this model.

3. DEDUCTION OF THE GENERAL WAVE EQUATION FOR ISOTROPIC, PERFECTLY ELASTIC MEDIA STARTING FROM THE THEORY OF CONTINUUM MECHANICS AND ELASTICITY RESPECTIVELY

Usually, the mechanical behaviour of a solid object is described by the theory of continuum mechanics. To use this theory, we must consider the macroscopic dimensions (for example the wavelength of acoustic signals) to exceed the microscopic dimensions (for example the lattice constant of the crystal) by far.

A linearized mass law for description of such media is the generalized Hooke's law, that describes the relation between stress and deformation in a perfectly elastic body:

$$T_{ij} = E_{ijkl} \cdot D_{kl} \quad (1)$$

This means, that in every point of the body the elements of the symmetrical part of the deformation tensor \mathbf{D} (or Green's distortion tensor) are a linear function of the stress tensor \mathbf{T} . E_{ijkl} is the elastic stiffness tensor, which is of fourth order. From the conservation of moment of momentum, we know that \mathbf{T} is symmetrical. The form or eq. (1) can generally be deduced from a potential basing on a consideration of the energy. It can be shown, that the originally 81 elements of \mathbf{E} may be reduced to 21 independent elements for a general *isotropic* body. When the elastic properties of an isotropic body do not depend on the orientation of the coordinate system, the relation stress-deformation may be described by two constants of elasticity:

$$T_{ij} = \lambda \theta \delta_{ij} + 2 \mu D_{ij} \quad (2)$$

μ and λ are called Lamé's constants. θ is the trace of the deformation tensor (dilatation in volume) and is called briefly dilatation.

Another essential element for the description of the elastic behaviour of solid bodies is the static condition of equilibrium:

$$f_i + \frac{\partial T_{ij}}{\partial x_j} = 0 \quad (3)$$

A volume unit is in equilibrium, when the sum of all body forces f_i (to be exact, here we should say force densities) is zero. The transition of this static condition of equilibrium to the dynamical analogue can be made by adding Newton's inertial forces to the applied forces, considering the *principle of d'Alembert*. Thus we get the equation of motion in its more general, linearized form:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = f_i + \frac{\partial T_{ij}}{\partial x_j} \quad (4)$$

Considering eq. (2), we yield \mathbf{T} in isotropic media:

$$\mathbf{T}_{ij} = \lambda \theta \delta_{ij} + 2\mu \mathbf{D}_{ij} = \lambda D_{kk} \delta_{ij} + 2\mu D_{ij} \quad (5)$$

After introducing the displacements u (see ch. 1.)

$$\mathbf{D}_{ij} = \frac{1}{2} \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \quad (6)$$

we get from eq. (5)

$$\rho \frac{\partial^2 u_i}{\partial t^2} = f_i + \frac{\partial}{\partial x_j} \left[\lambda \cdot \frac{\partial u_k}{\partial x_k} \cdot \delta_{ij} + \mu \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \right] \quad (7)$$

In homogeneous media, λ and μ do not depend on the position in the medium.

This (and only this) facilitates the following development:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = f_i + \lambda \frac{\partial^2 u_k}{\partial x_i \partial x_k} + \mu \frac{\partial^2 u_j}{\partial x_i \partial x_j} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \quad (8)$$

$$\rho \frac{\partial^2 u_i}{\partial t^2} = f_i + (\lambda + \mu) \frac{\partial \theta}{\partial x_i} + \mu \Delta u_i \quad \text{where} \quad \theta = \frac{\partial u_i}{\partial x_i} \quad (9)$$

Using $\Delta(\vec{u})_i = (\text{grad div } \vec{u} - \text{rot rot } \vec{u})_i$, we may generalize eq. (9) to any Cartesian system of coordinates:

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = \vec{f} + (\lambda + 2\mu) \text{grad div } \vec{u} - \mu \text{rot rot } \vec{u} \quad (10)$$

where $(\lambda + 2\mu)$ is equivalent to the isentropic bulk modulus. μ is called modulus of elasticity in shear.

4. DECOUPLING OF THE EQUATIONS OF MOTION

As we want to restrict ourselves further on to deformations, we are not interested in the f -term any more. It can be omitted without restriction of generality. Eq. (10) is also called fundamental equation of elastodynamics. In homogeneous, isotropic media that satisfy this equation, the right term may be split into two equations standing for two parts of motion. Mathematically, this is obtained by forming the divergence (div) and the curl (rot) of eq. (10).

First, the differentiation with respect to x_i shall be applied to eq. (10) (using θ of eq. (9)). This facilitates the elimination of the third summand using the relation ($\text{div rot } \vec{u} = 0$).

$$\rho \frac{\partial^2 \theta}{\partial t^2} = (\lambda + 2\mu) \Delta \theta \quad (11)$$

Hence follows

$$\frac{\partial^2 \theta}{\partial t^2} - \frac{\lambda + 2\mu}{\rho} \Delta \theta_p = 0 \quad (12)$$

Thus we get an inhomogeneous wave equation where only the compression part of the displacement appears. By applying the divergence, the curl terms have vanished. Observing the motion of single particles caused by this part of wave (fig. 1), it can be realized that they move parallel to the direction of wave travel; this is called a *plane longitudinal wave*. The factor on the left hand side $(\lambda + 2\mu)/\rho$ has the dimension of a squared velocity. It describes the propagation velocity of the compression wave:

$$v_p = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{\epsilon (1 - \sigma)}{\rho (1 + \sigma) (1 - 2\sigma)}} \quad (13)$$

Often, it is convenient to define additional elastic constants, such as Young's modulus ϵ and Poisson's ratio σ , which are merely combinations of the Lamé's constants [1].

Secondly, we want to apply the curl to eq. (9) for eliminating the divergence terms using the identity (*rot grad $\theta = 0$*):

$$\left[\rho \frac{\partial^2}{\partial t^2} \text{rot } \vec{u} = \text{rot } \vec{f} + (\lambda + \mu) \text{rot grad } \theta_s + \mu \Delta \text{rot } \vec{u} \right]_i \quad (14)$$

Substituting $(\text{rot } \vec{u})_i = 2 \vec{\phi}$ we get from eq. (12):

$$\frac{\partial^2 \vec{\phi}}{\partial t^2} - \frac{\mu}{\rho} \Delta \vec{\phi} = \frac{1}{2\rho} \text{rot } \vec{f} \quad (15)$$

As this is a source-free expression, eq. (15) represents an inhomogeneous wave equation for the pure shear part. The corresponding parts of the wave cause a particle movement perpendicular to the direction of wave travel (fig. 1b). For that reason, we can talk of *transverse waves*. The term μ/ρ stands for the square of the propagation velocity of this shear wave:

$$v_s = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{\epsilon}{2\rho(1+\sigma)}} \quad (16)$$

In homogeneous media, both kinds of waves propagate independently, P- and S-waves are non-interacting. This non-interaction can only be observed in homogeneous media. When the elastic properties or the density of the medium change, P-S-coupling immediately appears. The best example for this phenomenon are interfaces between two homogeneous media. At moderate changes of the elastic properties, the P-S-coupling can be neglected. Carrying out measurements on green concrete, we have to ask ourselves, if P- and S-waves propagate non-interacting, because only in this case the above-mentioned simplifications may be used.

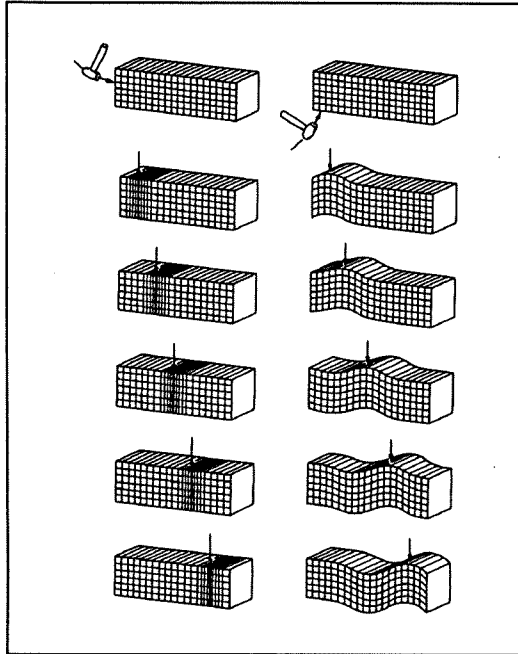


Fig.1: Propagation of an US-pulse in a bar of elastic material [6]:
 a) compression wave; b) shear wave.

In liquids, the shear modulus μ doesn't exist, neither does v_s . This means that in liquids no shear waves propagate ($v_s=0$, $\mu=0$). Very young concrete has to be considered a very viscous liquid. Only little by little (this means during some hours after mixing the concrete), a shear wave is forming. For solid concrete, we can assume in first approximation $\mu \sim \lambda$ and $v_s = 1/\sqrt{3} v_p$. During the hardening of concrete, the shear wave velocity will move between these two values.

5. SOLUTION OF THE WAVE EQUATION

For solving eq. (12) and (15), we choose a function of x and t . We assume plane waves which facilitate propagation in x -direction. Then, we can find two other functions depending on the coordinates y or, respectively z . As the differential equation is a linear one, we may use the superposition principle, which means that it is possible to assemble any arbitrary function out of the three independent solutions.

Inserting the velocities v_p and v_s , and assuming $\Phi = \Phi(x,t)$, where $\Phi \in \{\bar{\phi}, \theta\}$ we get a form of the equation of motion for the compression (θ) and for the shear ($\bar{\phi}$) wave part using eq. (12) or eq. (15), respectively:

$$\frac{\partial^2 \bar{\phi}}{\partial t^2} - v_s^2 \bar{\phi} = 0 \quad (17)$$

$$\frac{\partial^2 \theta}{\partial t^2} - v_p^2 \Delta \theta = 0 \quad (18)$$

These two equations represent simple three-dimensional wave equations. Each one is associated with an elastic wave travelling with velocity v_p or, respectively v_s . The equation for $\bar{\phi}$ (according to eq. (15)) only depends on the curls in the body (as $\text{div } \bar{\phi} = 0$). This shear wave consists of two parts polarized in two directions (y, z) perpendicular to the direction of propagation (x) named SV- and SH-wave. On the other hand, the wave θ , travelling in x -direction (according to eq. (12) and (9) respectively), implies changes of the volume (compressions and dilatations).

Thus, the solutions are customarily written as

$$\theta = \theta_0 \cdot e^{i(k_p x - \omega t)} \quad (19)$$

$$\vec{\Phi} = \vec{\Phi}_0 \cdot e^{i(k_s x - \omega t)} \quad (20)$$

where k is the wave number. It is related to the velocity by the dispersion relation (ω = angular frequency of the wave):

$$\omega_p = k_p \cdot v_p \quad (21)$$

$$\omega_s = k_s \cdot v_s \quad (22)$$

The dependence of the wave velocities on the two elastic parameters (eqs. 13 and 16) is usually already sufficient for computing ϵ and σ (when the density ρ as well as v_p and v_s are well-known). In concrete, however, this way is associated with difficulties in the testing technique. The attenuation of body waves in concrete is so high, that one is urged to work with transducers of high sensitivity. One of their disadvantages is that they show long post-pulse oscillations. So the coda¹ of the P-wave often superimposes the incidence of the shear wave. Additionally, stimulation and transmission of shear forces are by far more difficult than of compression forces. As a result of these two aspects, it is generally very difficult to determine the S-wave arrival times and with this v_s . Indeed, the seismic plots of US-measurements in concrete also contain another type of wave that is attenuated much less; surface waves.

¹lat.: cauda=tail; in seismology: waves occurring in the seismic plot immediately after the incidence of a wave type and being attached to it (source pulse, conversions, reflexions, and so on).

6. SURFACE WAVES

For the special case of the formerly regarded waveforms meeting a free boundary, another waveform exists. Formally it can be considered as a combination of P- and S-components [4]: *Rayleigh-waves*. In opposition to *Love-waves* that exist as a special kind of the so-called surface waves in layered media only, *Rayleigh-waves* can also be found in the homogeneous half-space.

For finding the solutions of eqs. (17)/(18) that correspond with such waves, we orientate the z-axis parallel to the surface normal. Hereby, $z < 0$ means the inside of the body. Thus, for $z=0$ the components T_{13} , T_{23} , and T_{33} of the stress tensor (= stress vector applied to the surface) disappear, because at free surfaces, neglecting atmospheric pressure, no stress occurs. Furthermore, we assume a plane and infinite boundary. For a wave propagating in x-direction, whose amplitude depends on the distance from the boundary (z), we use the following equation for Φ :

$$\Phi = e^{i(kx - \omega t)} \Psi(z) \quad (23)$$

Using this equation in eq. (17)/(18), we receive a conditional equation for $\Psi(z)$:

$$\frac{\partial^2}{\partial z^2} \Psi = \kappa_{p/s}^2 \Psi \quad \text{where} \quad \kappa_{p/s} = \sqrt{\left(k^2 - \frac{\omega^2}{v_{p/s}^2}\right)} \quad (24)$$

In case of the term ω^2/v^2 in this equation being higher than the square of the wave number, this means that at an increasing period of the surface waves, we get a transition to the free oscillations of the body. $\kappa^2 < 0$ and κ becomes imaginary in this case. Here, we only want to consider a real κ , this makes us express $\Psi(z)$ by the following equation:

$$\Psi(z) = a e^{\kappa z} + b e^{-\kappa z} \quad (25)$$

The solution with a negative exponent κz is associated with a wave of increasing amplitude with respect to the depth. Physically this is not significant and shall not be considered further on. Now we get the solutions of the wave equation for Rayleigh-waves using eq. (23) and eq. (25):

$$\Phi = \text{const} \cdot e^{i(\kappa x - \omega t)} \cdot e^{-\kappa z} \quad (26)$$

Having a view at this equation, we can see that the wave first of all propagates along the free surface ($z=0$), because its amplitude decays exponentially with respect to the depth (z). This wave is polarized elliptically, where the phase shift of the vertical and horizontal components is 90° [4]. The amplitude ratio of these two components runs up to $1\frac{1}{2}$ at the surface. Fig. 2 illustrates the elliptical retrograde motion of the Rayleigh-wave in a homogeneous, ideal-elastic medium with a free surface.

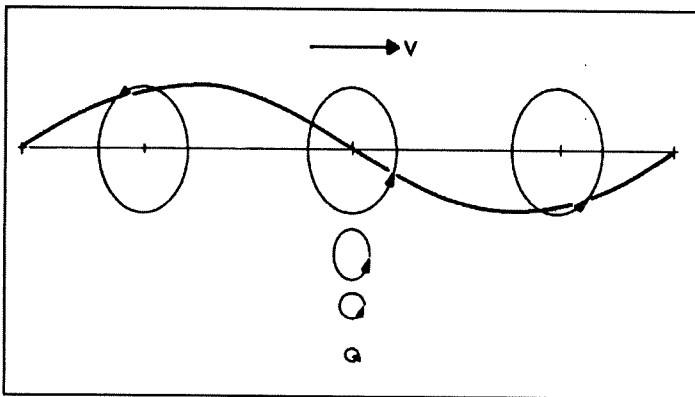


Fig. 2: Particle motion of a Rayleigh-wave in a homogeneous, ideal-elastic medium with a free surface [6].

7. RELATION BETWEEN THE DIFFERENT WAVE VELOCITIES

The displacement vector of the surface wave consists of the sum of the vectors of the P- and S-wave part, according to eqs. (17)/(18). For body waves in infinite homogeneous media, we have shown in ch. 4 the possibility of decoupling the two wave components. Because of the boundary conditions, a separation of two independent parts of the surface waves is unfortunately impossible. For that reason, to get the real displacement vector \vec{u} , a linear combination of the vectors \vec{u}_p and \vec{u}_s has to be determined. Nevertheless, the boundary conditions (no stress on the surface) may help us to find a conditional equation. We use plane waves for the displacement vector, analogous with the splitting to P- and S-waves (P-, SV, and SH-waves respectively), where the wave front travels in x-direction and a possible dependence on z being considered according to eq. (25). Considering the components of the P-wave-part satisfying the condition $\text{rot } \vec{u}_p = 0$ as well as the S-wave-part satisfying $\text{div } \vec{u}_s = 0$, we get an implicit dispersion relation for the surface wave [1, 2], using the solution of eq. (26):

$$(k^2 + \kappa_s^2)^2 = 4k^2 \kappa_p \kappa_s \quad (27)$$

Inserting the right hand part of eq. (24), we get:

$$\left(2k^2 - \frac{\omega^2}{v_s^2}\right)^4 = 16 k^4 \left(k^2 - \frac{\omega^2}{v_s^2}\right) \left(k^2 - \frac{\omega^2}{v_p^2}\right) \quad (28)$$

Where k is the wave number and ω the angular frequency of the Rayleigh-wave. Unfortunately, unlike eqs. (21)/(22), eq. (28) cannot be resolved with respect to k. Indeed, using a linear dispersion relation, it is possible to make

some simplification. Assuming the linearization $\omega = \text{const} \cdot k$, we introduce

$$\omega = v_r \cdot k \quad (\text{where } v_r = \text{velocity of the Rayleigh-wave}) \quad (29)$$

Eq. (28) takes the form

$$\left(2 - \frac{v_r^2}{v_s^2}\right)^4 = 16 \left(1 - \frac{v_r^2}{v_s^2}\right) \left(1 - \frac{v_r^2}{v_p^2}\right) \quad (30)$$

In eq. (31), we've found a relation between the three wave velocities.

As mentioned in ch. 5, a measurement of the S-wave-velocity in concrete is generally very difficult. Thus, it is desirable to find a way to compute the elastic parameters (σ , ϵ , and so on) by merely making measurements of the velocities v_p and v_r , which are easy to determine². This means, using above equation, we aim at computing v_s or at computing directly the elastic parameters ϵ and σ from v_p and v_r (see ch. 8). First of all however, examining the foregoing relationship, we should find out, how strong the mutual interdependence is, which values we get for this equation, and what the function looks like in reality.

Transforming eq. (30), we get a polynomial of 4th degree in v_s^2 . As the explicit solution takes some pages (for example by the numeric-software *Mathematica*), we used eq. (30) in a slightly modified form for the determination of v_s by computing the zeros of this polynomial:

² There is no dispersion in unlayered hardened concrete. Therefore it is possible to determine the velocity of the Rayleigh wave from the group velocity directly.

$$\left(2 - \frac{v_r^2}{v_s^2}\right)^2 - \left(4 \sqrt{1 - \frac{v_r^2}{v_p^2}} \sqrt{1 - \frac{v_r^2}{v_s^2}}\right) = 0 \quad (31)$$

The interdependence of v_s from v_p and v_r can be demonstrated by varying v_p as well as v_r , according to the standard values in concrete (attention: singularities!). This is shown in the following two figures as results of exact numerical calculations. In fig. 4a we used a fixed v_p (4000 m/s), in fig. 4b a fixed v_r (2200 m/s).

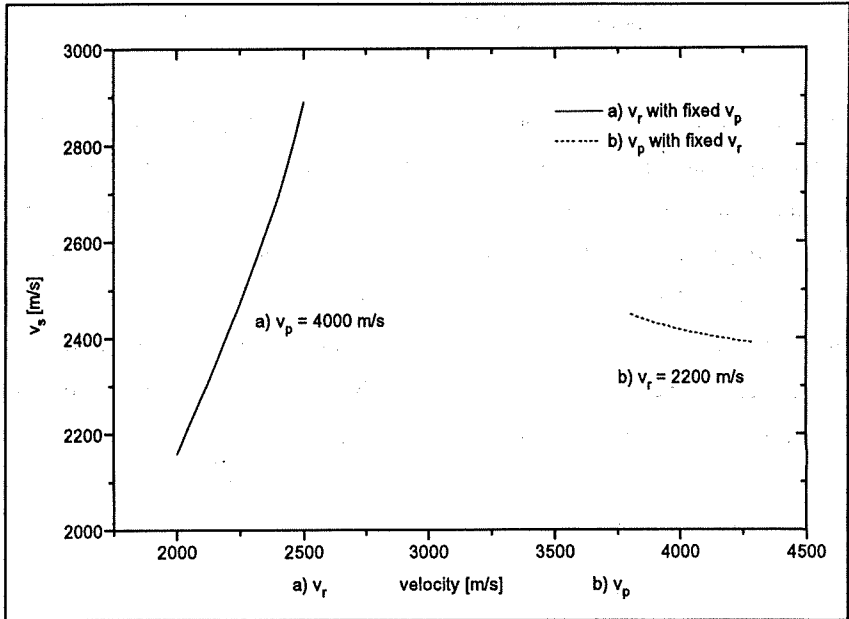


Fig.4: Dependence of the shear wave velocity v_s ,
a) with fixed v_p (4000 m/s), b) with fixed v_r (2200 m/s).

A stronger interdependence of v_s (and of ϵ , σ , ...) upon v_r than upon v_p can easily be realized. In fact, as shown in ch. 6, we should say it the other way-round: v_r depends more on v_s than on v_p .

This relationship between v_p , v_s and v_r according to eq. (31) may also be presented three-dimensional or as isoline-plot. For this, eq. (31) was transformed in several steps to receive an explicit equation for v_p :

$$v_p^2 = \frac{16 v_r^2 (1 - \xi^2)}{16(1 - \xi^2) - (2 - \xi^2)^4} \quad \text{where} \quad \xi = \frac{v_r}{v_s} < 1 \quad (32)$$

For all values of $v_r \geq v_s$ the denominator becomes negative and with it v_p imaginary. Only the areas of the function where the Rayleigh wave velocity is smaller than the shear wave velocity are physically significant. In fig. 5 and 6 the interdependence of the P-wave velocity from v_s and v_r is illustrated. Fig. 6 gives a general survey of the function course. Fig. 5 represents the same function in an isoline plot projected on v_r/v_s -plane. Only the v_r/v_s -values left of the median line are important. Additionally, because of eq.(13) and eq.(16) where $\lambda \geq 0$ the area where $v_p < \sqrt{2} v_s$ has to be excluded. This graph provides a quick outlook onto the possible v_r/v_s -velocities for a preset P-wave velocity (see fig. 4a for 4000 m/s).

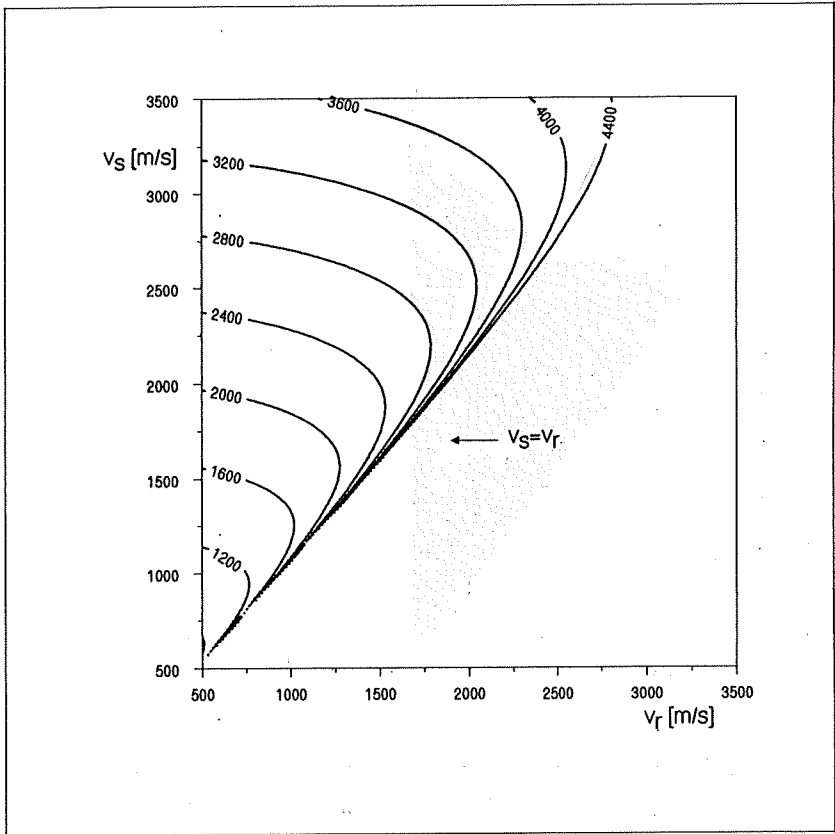


Fig.5: Interdependence of the three wave velocities (isoline plot).

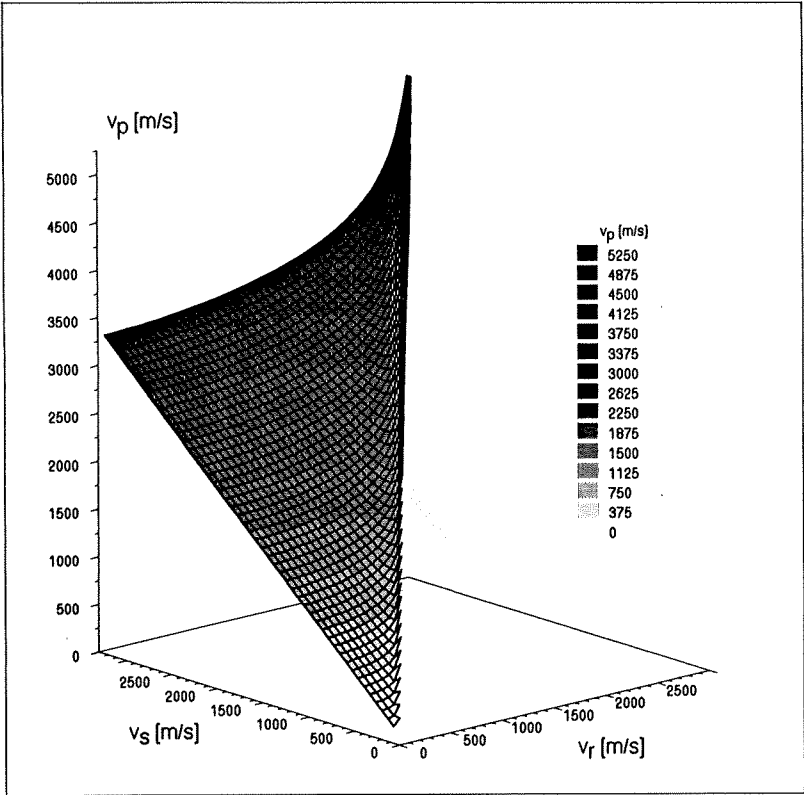


Fig.6: Interdependence of the three wave velocities (3D-plot).

The function is physically significant only for the area:

$$v_s > v_r > 0 \quad \text{and} \quad v_p > \sqrt{2} v_s \quad !$$

8. DETERMINATION OF THE ELASTIC PARAMETERS

8.1. Poisson's ratio

Now we are able to identify a proportional factor ξ that gives us the relation between the surface and the body wave velocity.

Using eq. (32), we assume:

$$v_r = \xi v_s \quad (33)$$

Using $\omega = v_r k \xi$ of eq. (29) in eq. (31), dissolving the brackets, and reducing the equation by k^8 , we get:

$$\xi^6 - 8\xi^4 + 8\xi^2 \left(3 - 2\frac{v_s^2}{v_p^2} \right) - 16 \left(1 - \frac{v_s^2}{v_p^2} \right) = 0 \quad (34)$$

Eq. (34) shows, that ξ only depends on v_p and v_s or, respectively directly on the ratio v_s/v_p , which is characteristic for every medium. On the other hand, this ratio only depends on Poisson's ratio [1]:

$$\frac{v_s}{v_p} = \sqrt{\frac{1-2\sigma}{2(1-\sigma)}} \quad (35)$$

Fig. 7 illustrates this relation. The parameter ξ is positive and real. Besides, it has to be $\xi < 1$ (for κ_s and κ_p being real). Eq. (34) only has one root that satisfies these conditions. So we get exactly one ξ for each given v_s/v_p -value. The direct relation between ξ and σ results of eq. (35) using eq. (34):

$$\xi^6 - 8\xi^4 + 8\xi^2 \left(3 - 2 \frac{1-2\sigma}{2(1-\sigma)} \right) - 16 \left(1 - \frac{1-2\sigma}{2(1-\sigma)} \right) = 0 \quad (36)$$

As σ must be smaller than 0.5 ($=\sigma_{\text{rubber}}$), from eq.(35) follows that the ratio v_s/v_p may only have values between 0.874 and 0.955 (shown in fig. 7).

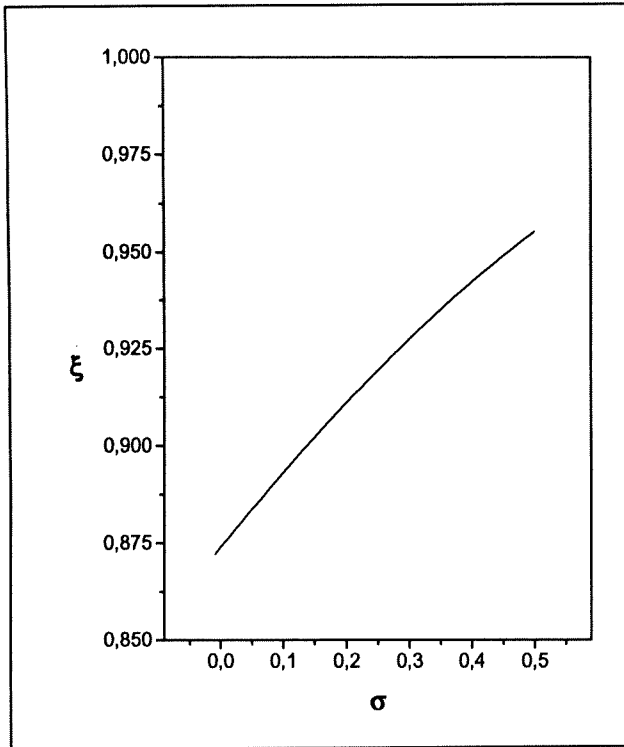


Fig.7: Interdependence of the parameter ξ upon σ (v_p, v_s).

To facilitate the calculation of Poisson's ratio experimentally directly from the wave velocities, we transformed eq. (35) to get the following equation for σ :

$$\sigma = \frac{1}{2} \cdot \frac{1 - 2 \frac{v_s^2}{v_p^2}}{1 - \frac{v_s^2}{v_p^2}} \quad (37)$$

Using eq. (31) we can eliminate v_s in this expression, so we get a function $\sigma = f(v_p, v_r)$.

8.2. Young's modulus

Thus it is possible to calculate directly Poisson's ratio using eq. (31) (v_s being computed numerically) and eq. (37). If in addition Young's modulus has to be determined, knowing the density of the medium ρ , we can use eq. (13) and get:

$$\epsilon = v_p^2 \rho \cdot \frac{(1+\sigma)(1-2\sigma)}{1-\sigma} \quad (38)$$

9. SEISMIC WAVES IN INELASTIC MEDIA

The inelastic behaviour of a medium (deviation from the perfect elasticity) may be described by the generalized mass law by Nakamura [4]:

$$\left(a + b \frac{\partial}{\partial t}\right) T_{ik} = \delta_{ik} \left(\lambda' + \lambda \frac{\partial}{\partial t}\right) D_{ii} + 2 \left(\mu' + \mu \frac{\partial}{\partial t}\right) D_{ik} \quad (39)$$

That has been deduced from a general after-effect law for deformation, where after-effect functions occur. The momentary state is, among others, influenced by the elastic antecedents; this is called after-effect. In visco-elastic media, it is for example necessary, to include the time history of the stress and deformation tensor. In this case, plastic deformation will not be considered.

Eq. (39) is valid for all special cases of linear elastic mass laws with maximum one derivative with respect to time. So the parameters a , b , and so on, can be determined. To be sure, all these coefficients for standard-models are specified in literature, but for the Bingham model this is rather complicated. Up to now, we couldn't find corresponding information. Usually, the determination of the occurring coefficients involves with great efforts.

It is even more difficult to state the wave equation of a wave propagating through an inelastic body. Considering harmonic waves with the time dependency $e^{i\omega t}$, we get from eq. (39) for the stationary state:

$$T_{ik} = \delta_{ik} \left(\frac{\lambda' - i\omega\lambda}{a - i\omega b}\right) D_{ii} + 2 \left(\frac{\mu' - i\omega\mu}{a - i\omega b}\right) D_{ik} \quad (40)$$

This mass law is valid for periodical events with angular frequency ω and corresponds exactly to Hooke's law in case of ideal elasticity. By letting converge the expressions in brackets towards Lamé's constants, this can be ascertained.

How are, for example, plane waves affected by the complex terms of eq. (40)? Considering a P-wave travelling in x-direction, we get the following solution according to eqs. (19)/(20) ($A \neq 0$, where A =amplitude in x-direction):

$$\vec{u} = \frac{\vec{x}}{|\vec{x}|} A e^{i(\vec{k}_p \cdot \vec{x} - \omega t)} \quad (41)$$

where again eqs. (13) and (21) are used:

$$k_p = \omega \sqrt{\frac{\rho}{\lambda + 2\mu}} \quad (42)$$

As in the inelastic case, λ and μ become complex, also the wave number k_p becomes complex:

$$k_p = \omega \sqrt{\frac{\rho (a - i\omega b)}{\lambda' + 2\mu' - i\omega(\lambda + 2\mu)}} = \alpha + i\beta \quad (43)$$

The displacement vector has the form:

$$\vec{u} = \vec{x} A e^{(-\beta x + i(\alpha x - \omega t))} \quad (44)$$

\vec{u} shows an exponential amplitude decay in direction of travel, where the amplitude has decayed to $1/e$ after the travel path τ :

$$\tau = \frac{1}{\beta} = \frac{1}{\text{Im}(k_p)} > 0 \quad (45)$$

τ is called *penetration depth*. In this case, the phase velocity depends on frequency:

$$v_p(\omega) = \frac{\omega}{\alpha} = \frac{\omega}{\text{Re}(k_p)} \quad (46)$$

Thus, it becomes clear that under above-mentioned conditions for visco-elastic media, frequency-dependent absorption and dispersion occur. This means, acoustic waves are subject to a change of their waveform during their propagation.

10. CONCLUSIONS

Starting from the theory of elasticity, it is possible to deduce an expression for the mutual interdependence of the velocity of body (P- or S-) and surface waves. The appropriate relation has been displayed mathematically as well as graphically for creating an impression of the physically significant values. The deduced relations facilitate the computing of the elastic parameters (Young's modulus and Poisson's ratio) during the experiment directly from the velocities of compression wave and Rayleigh-wave.

The condition for developing the equations and for above proceeding is indeed the perfectly elastic behaviour of the tested medium when an acoustic wave propagates through it. Now we know exactly now at which points of the deduction this condition is needed. In addition, first approaches to consider inelasticity have been presented. It is hard to estimate the strength of this effect in green concrete and whether it is eventually possible to find a correction term for above relation. While hardening, concrete goes through all states between liquid and solid.

At present, we're performing some experiments, where waveforms of an US-broadband pulses during the hardening of concrete are being analysed. We managed to verify a strong change of the waveform of the acoustic waves with respect to amplitude and frequency in course of the experiment. This points to the above-mentioned absorption and dispersion effects. Further on it will be investigated, whether it is possible to quantify these effects by a dispersion analysis.

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